Random Fields: Modeling Soil Properties

in *Stochastic Analysis and Inverse Modeling*

Presented by

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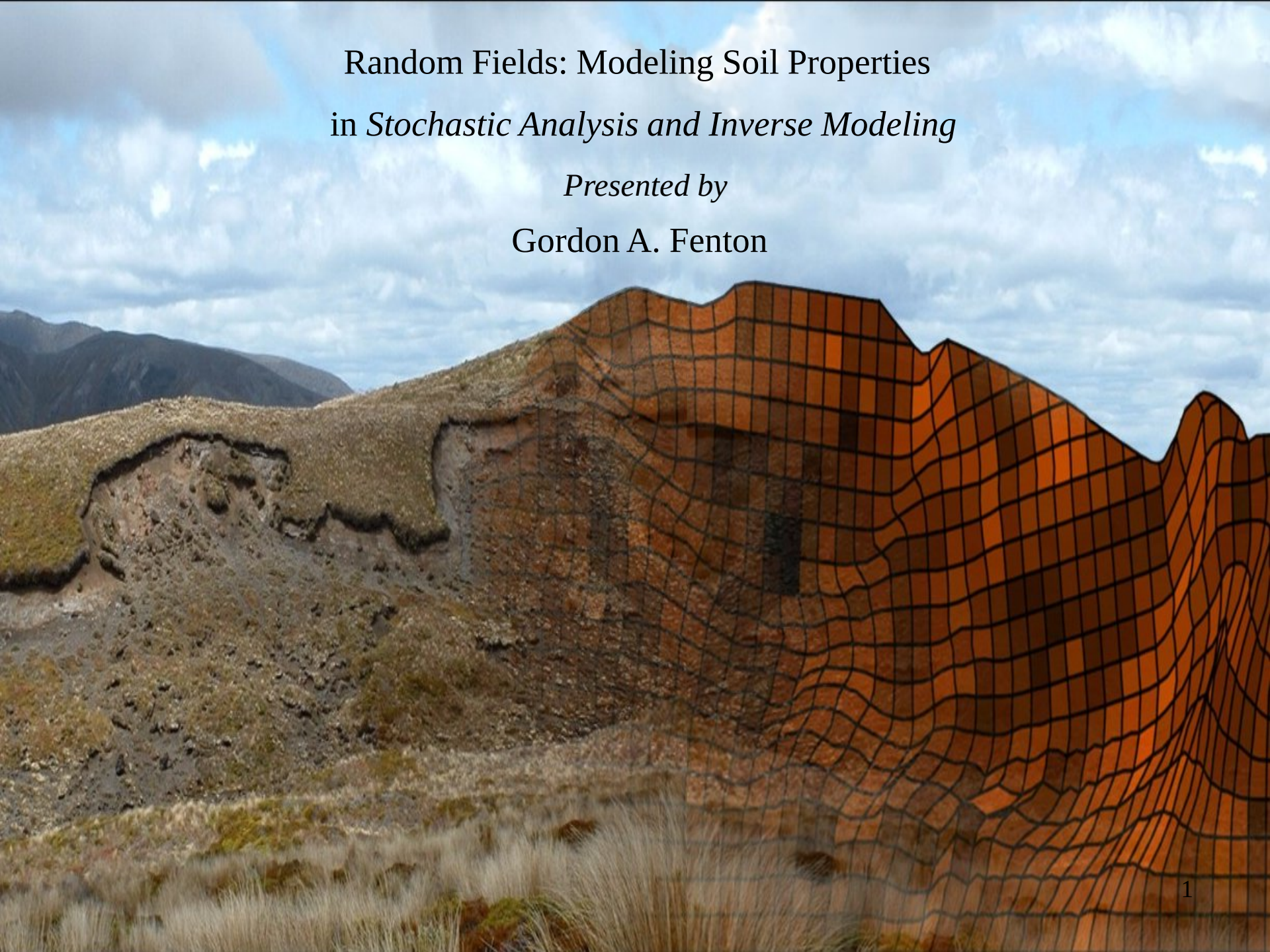


Introduction

In this lecture we are going to discuss

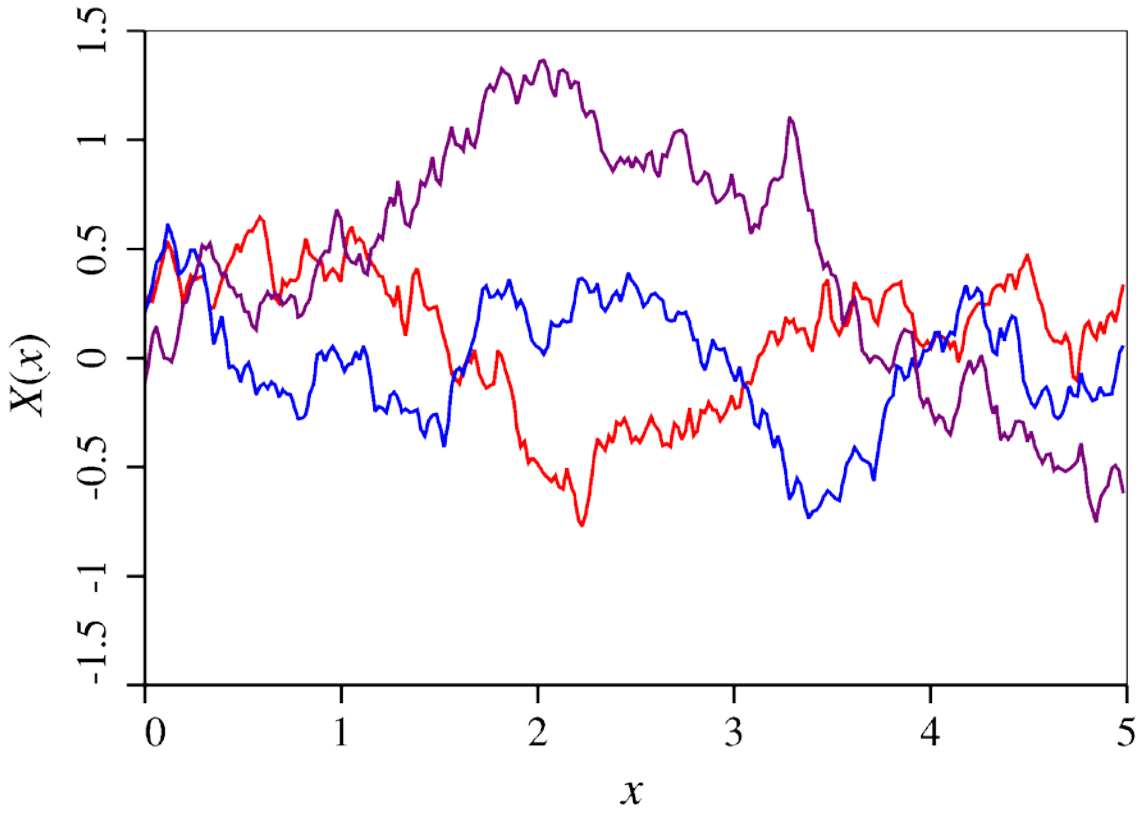
1. types of random fields,
2. characteristics of random fields
    a) mean – stationary vs non-stationary
    b) variance
    c) covariance function
    d) variance function

Motivation

- It is well known that soils are spatially variable.
- To investigate how spatial variability affects geotechnical behaviour, soils must be modeled as multi-dimensional multi-variate randomly varying processes (ideally 3-D). This allows questions such as "does the soil strength depend on the weakest link, or on an average?" and "how does spatial variability affect the probability of failure?" to be investigated.
- Random field theory provides a basis for such models.
- Random fields are characterized by
  - their mean, $\mu(\boldsymbol{x})$ (first moment)
  - their covariance structure, $C(\boldsymbol{x}', \boldsymbol{x}^*)$
  - higher order moments (difficult to estimate)

Random Fields

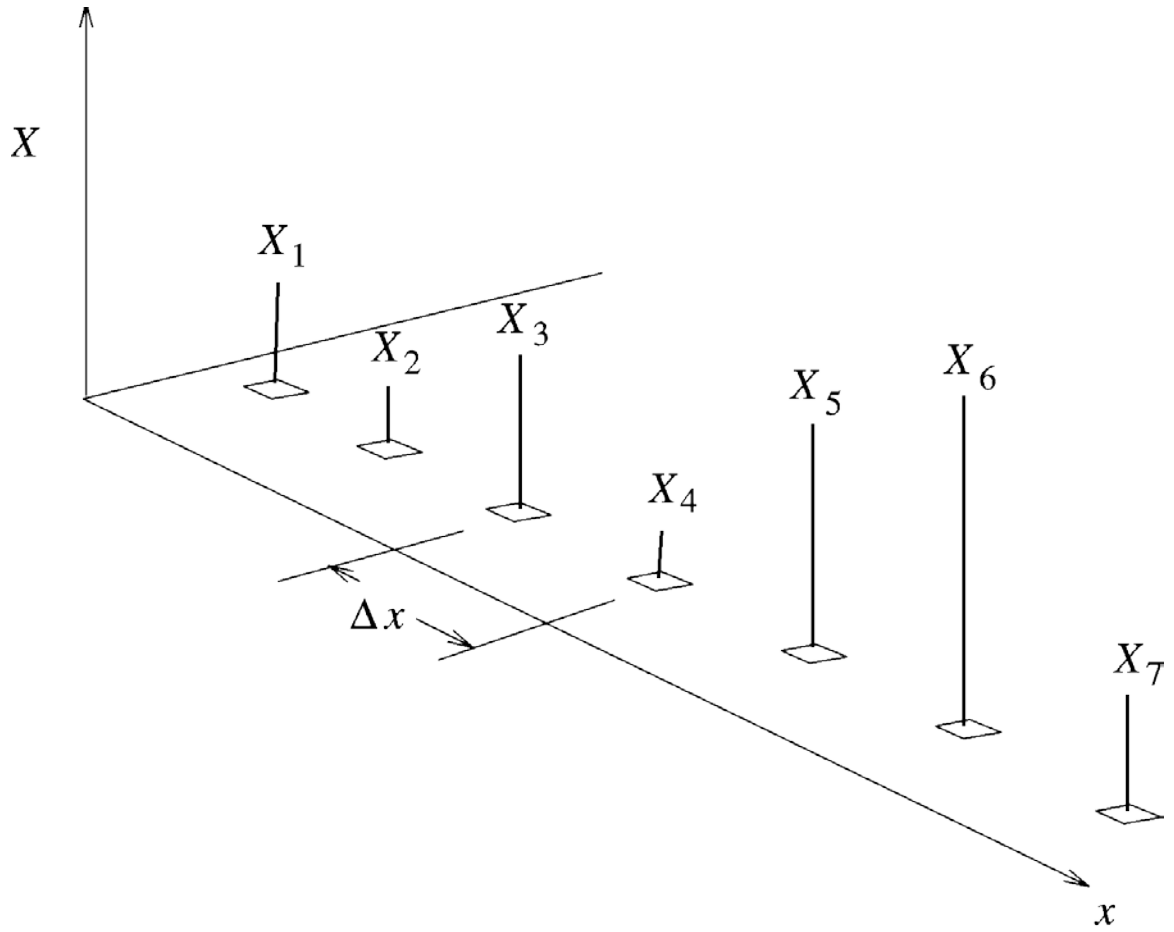
A random field is a collection of random variables, $X(\mathbf{x}_1)$, $X(\mathbf{x}_2)$, …, one for each point in the field. Each $X(\mathbf{x}_i)$ is uncertain until it is observed. We call an observation of $X(\mathbf{x}_i)$ a *realization*.



Three realizations of $X(x)$. The value of X at each x follows a distribution, $f_X(x)$, which might change with x.

Types of Random Fields

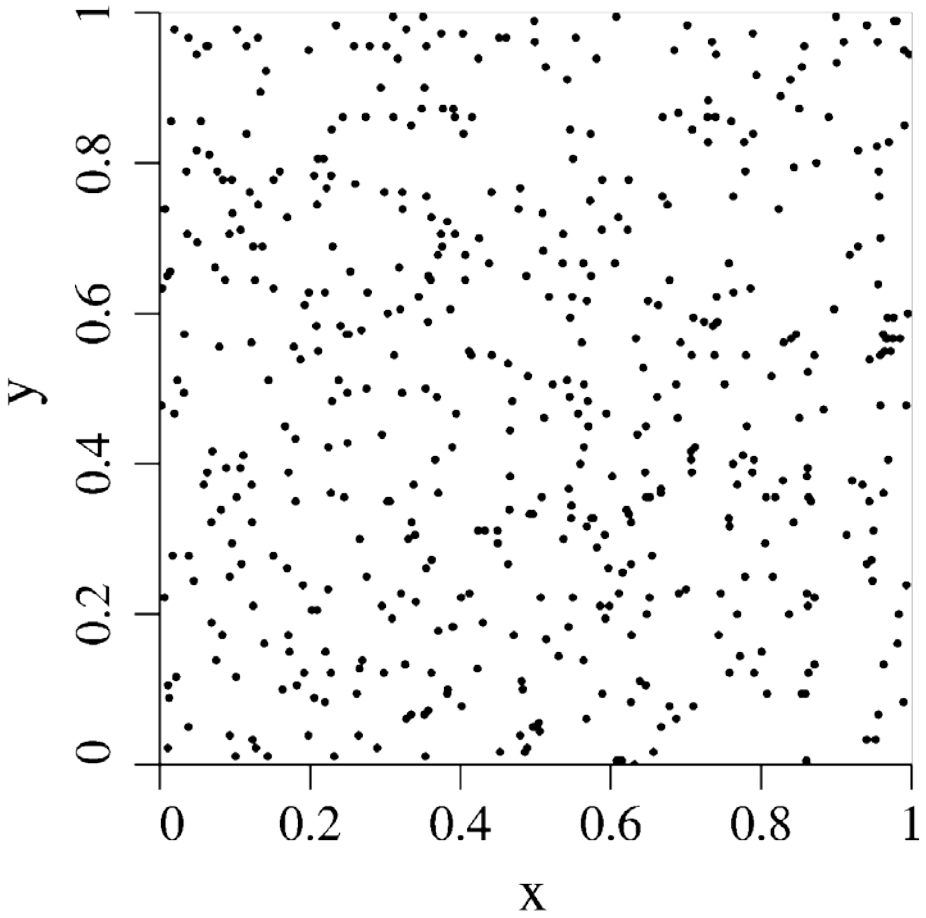
Discrete Random Fields: observations are taken at discrete points. E.g. soil samples.



Types of Random Fields

Random Point Processes: points occur randomly in space

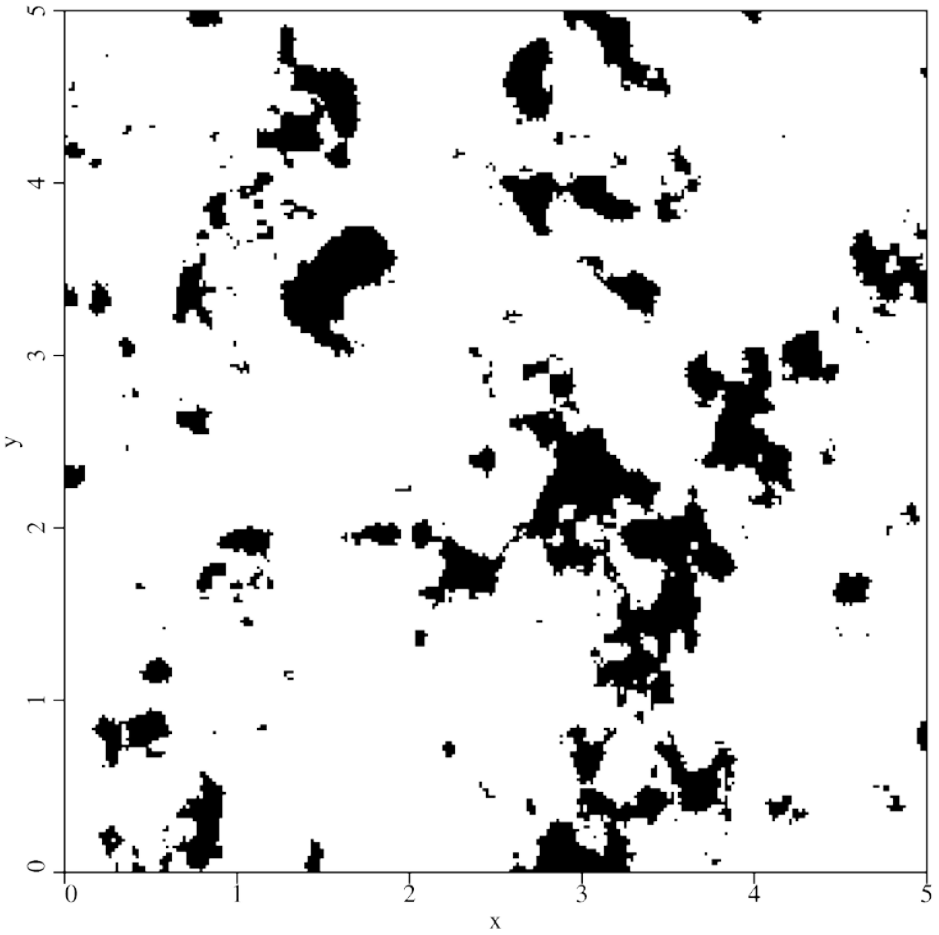
- useful for modeling occurrences of rock cracks (with other random processes for crack length and orientation).



Types of Random Fields

Binary Random Fields: a simple black and white field

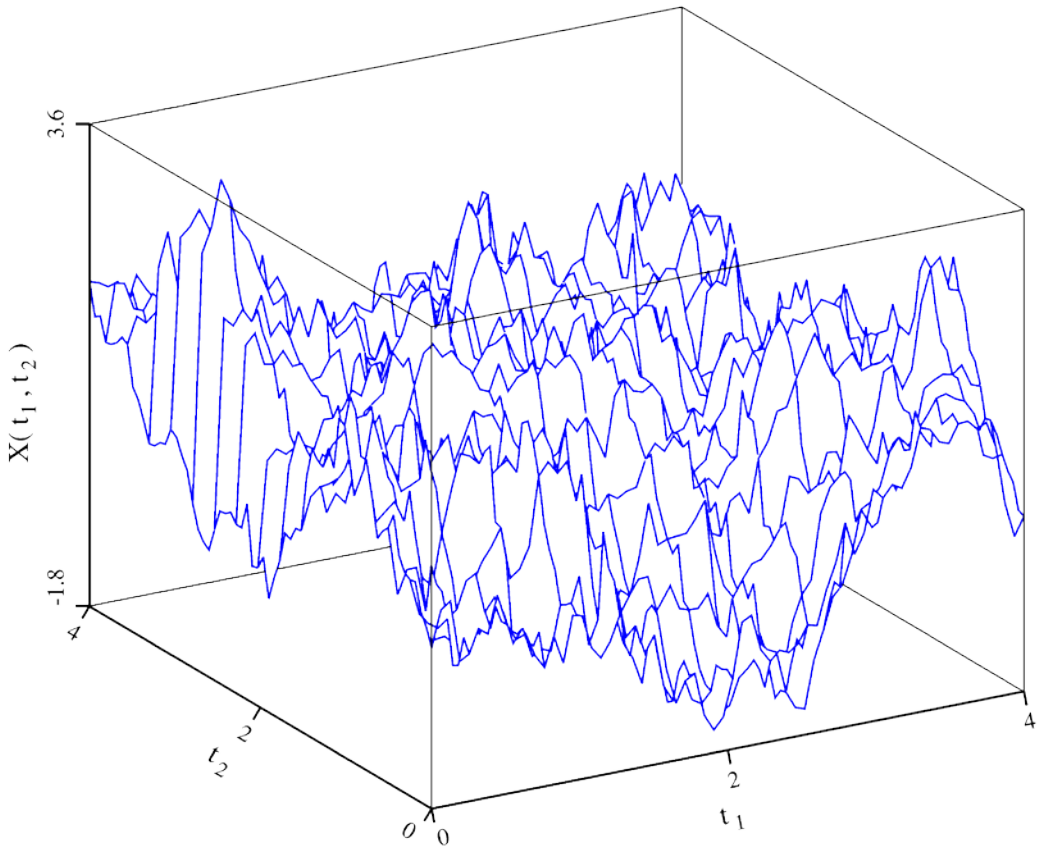
- useful for modeling liquefied regions, toxic waste regions, etc.



Types of Random Fields

Continuous Random Fields: the random quantity varies continuously over space.

- useful for soil property models (e.g. elastic modulus, cohesion, friction angle, etc.)



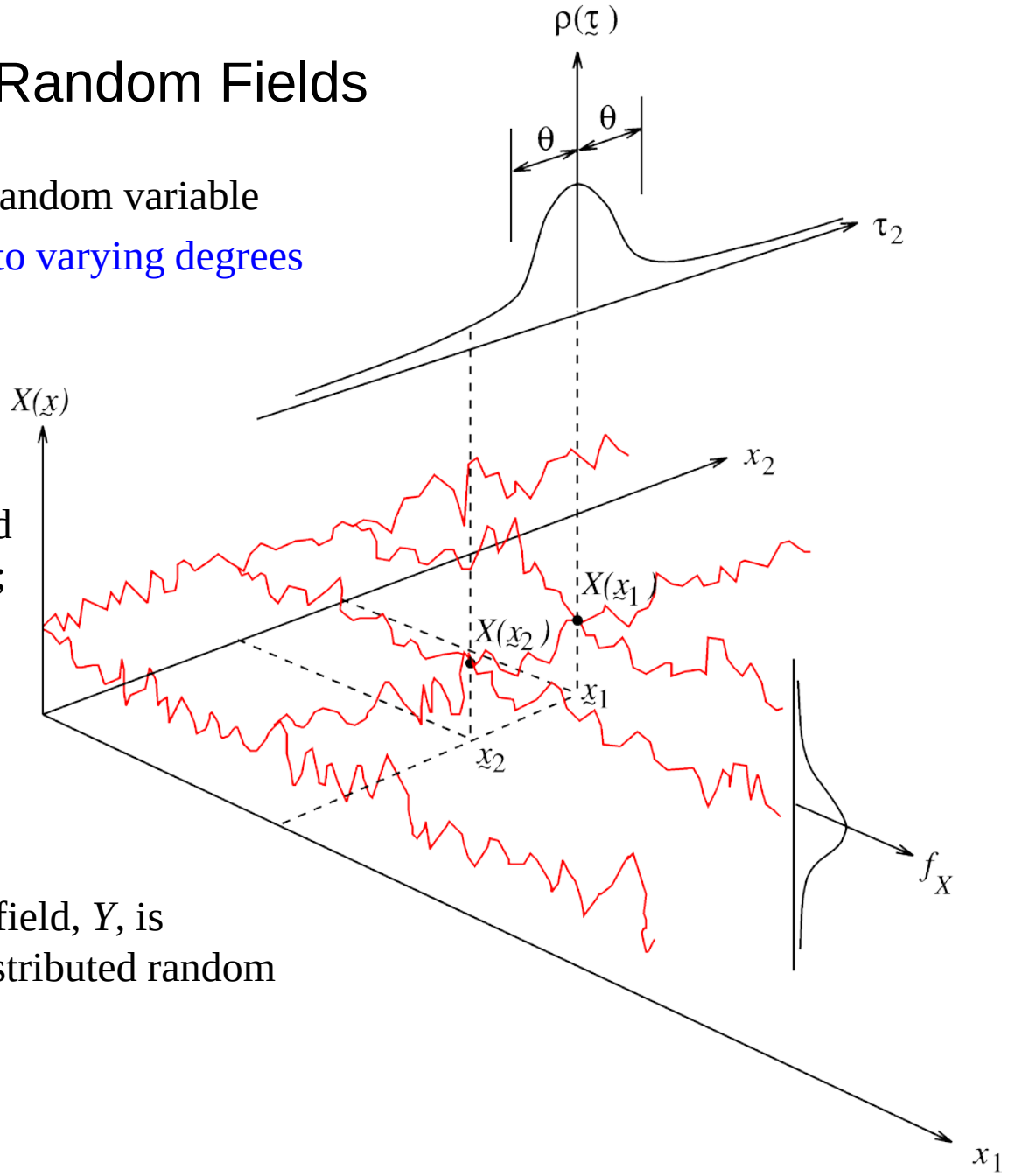
Random Fields

every point in the field (site) is a random variable

all points are mutually correlated to varying degrees

small θ → rough fields

large θ → smoother fields



If the random field is stationary and normally distributed, then we need;

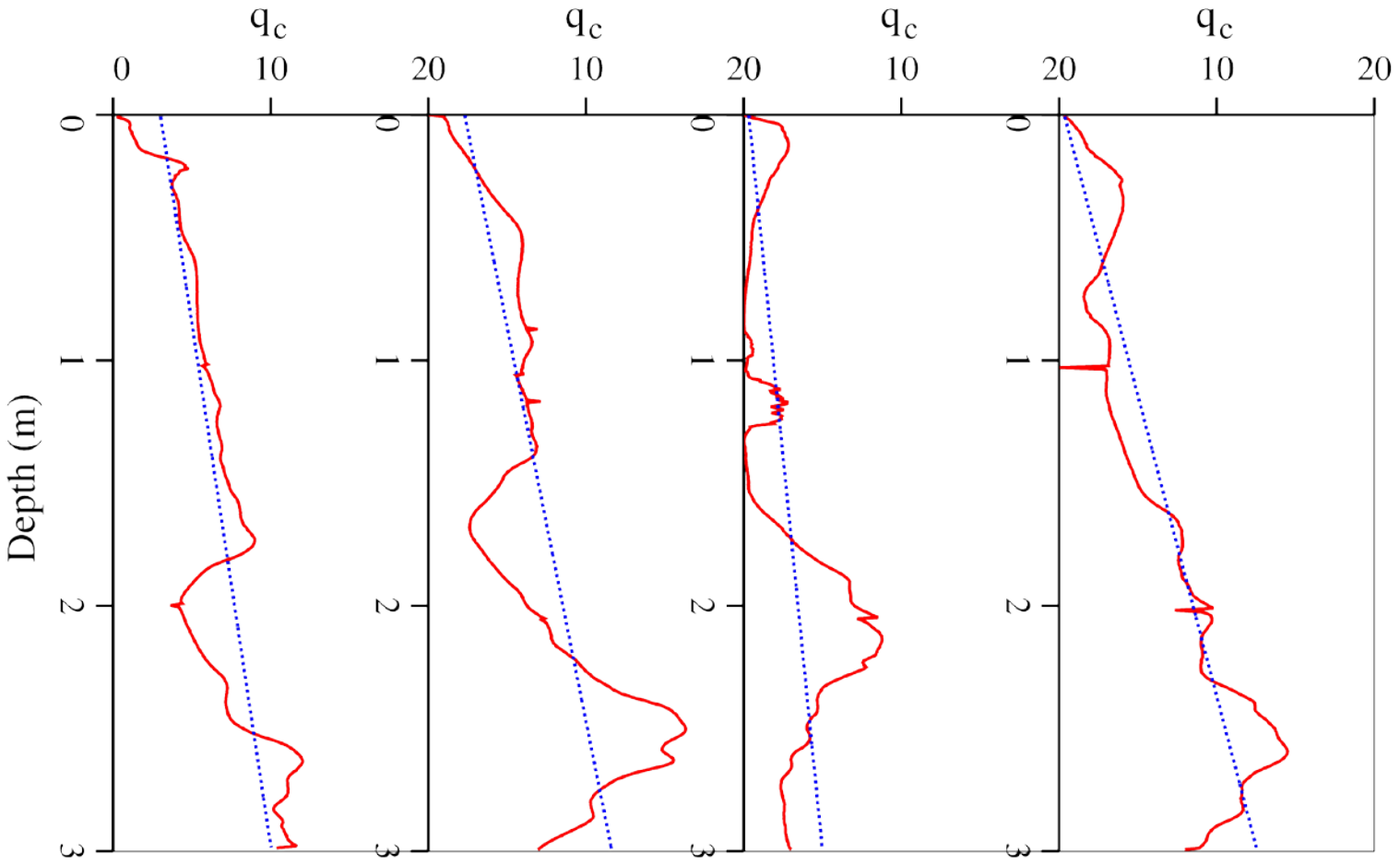
1. the field mean, μ_X ,
2. the field variance, σ_X^2 ,
3. the field correlation structure, commonly parameterized by the correlation length, θ_X

A lognormally distributed random field, *Y*, is easily obtained from a normally distributed random field, *X*, by the transformation

$$Y = e^X$$

Random Field Mean

In principle, the mean of a random field, $\mu(\mathbf{x})$, can vary with spatial position, **x.** For example, the mean shear strength of a soil often varies with depth. We call such a field *non-stationary*.



Random Field Mean

- Trends in the mean should only be used if;
  - the random field is representing the site at which the data were obtained, or
  - the trend is expected to occur at any site represented by the random field.
- Otherwise, the apparent trend should be viewed as merely part of the uncertainty to be characterized by the random field.

Random Field Variance

- The "point" variance of a random field can also be non-stationary, i.e., $\sigma(\mathbf{x})$ can vary with spatial position **x**.
- Estimating a non-stationary variance requires a lot of data, so variance is commonly considered to be *stationary*. That is, $\sigma(\mathbf{x}) = \sigma$ is assumed constant over the field. In this case, the entire data set can be used to estimate the common variance, σ^2.
- A random field with non-stationary mean and variance can always be converted to be *weakly stationary* (stationary in the mean and variance) through the linear transformation;

$$X'(\mathbf{x}) = \frac{X(\mathbf{x}) - \mu(\mathbf{x})}{\sigma(\mathbf{x})}$$

where now $X'(\mathbf{x})$ has stationary mean zero and unit variance.

Correlation Structure

Aside from the mean, μ_X , and the standard deviation, σ_X , a third important feature of a random field is its *degree of persistence*.

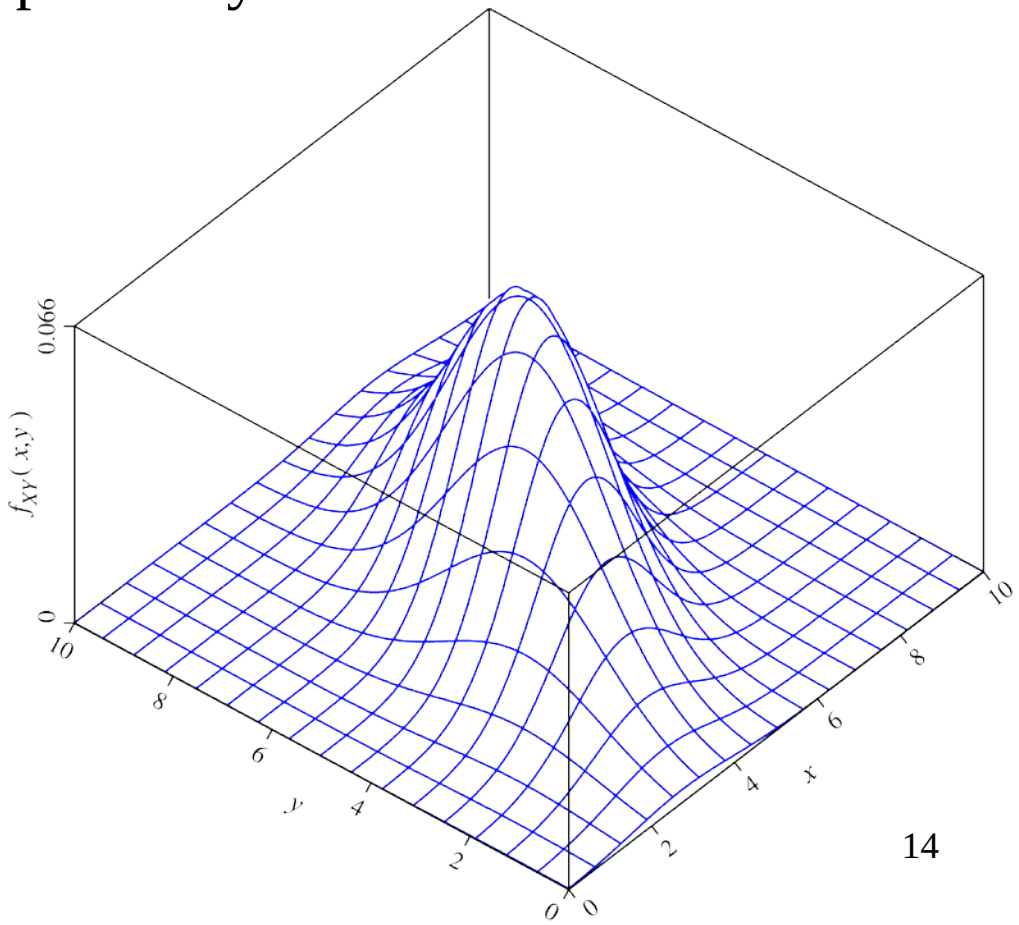
Suppose $X(2)$ is the friction angle at $x = 2$ m depth. Suppose further that at a particular site $X(2)$ is large. If the random field of friction angles at the site are persistent, then it is likely that $X(2.01)$ is also large.

We characterize this persistence by specifying a correlation coefficient acting between random variables: values near 1.0 imply a strong linear dependence between the random variables. E.g. as one increases, the other also increases.

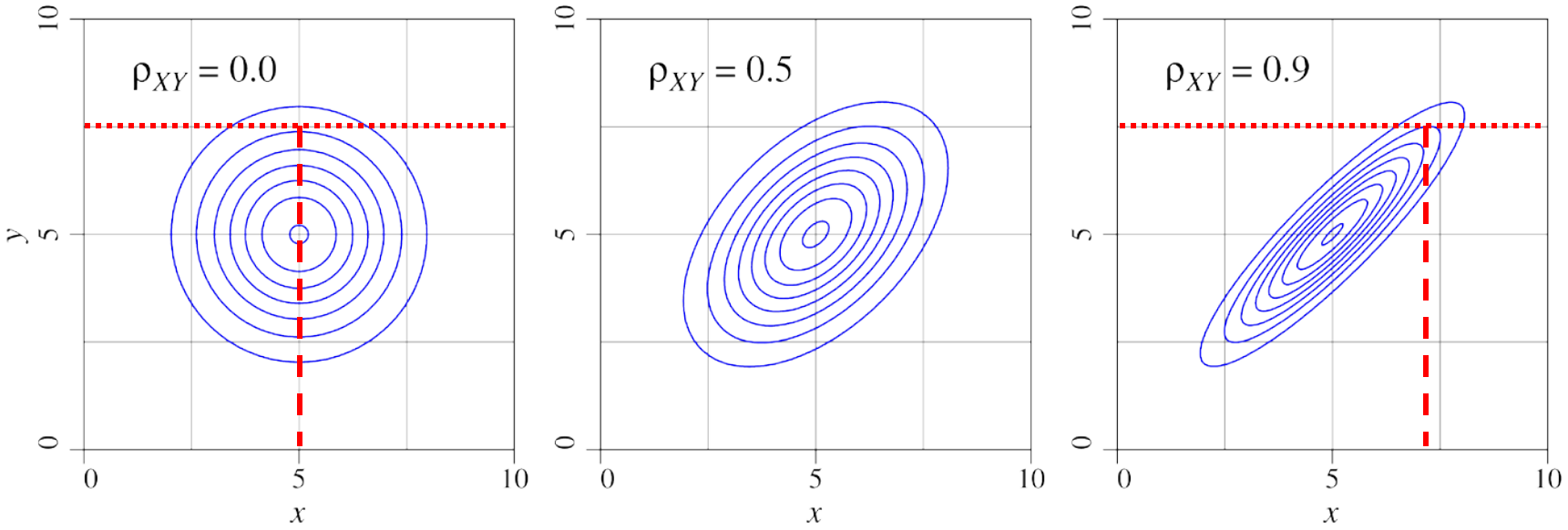
Correlation Structure

Consider two random variables, X and Y. The first measure of any dependence between them is the *degree of linear dependence*. This degree is captured by the correlation coefficient;

$$\rho_{XY} = \frac{\mathrm{Cov}[X,Y]}{\sigma_X \sigma_Y}$$



Correlation Structure



Bivariate probability density contours ($\mu = 5$ and $\sigma = 1.5$) for various correlation coefficients.

If $Y = a \pm bX$ then $\rho_{XY} = \pm 1$ ($a = 0$ and $b = 1$ in above plots).

Correlation Structure

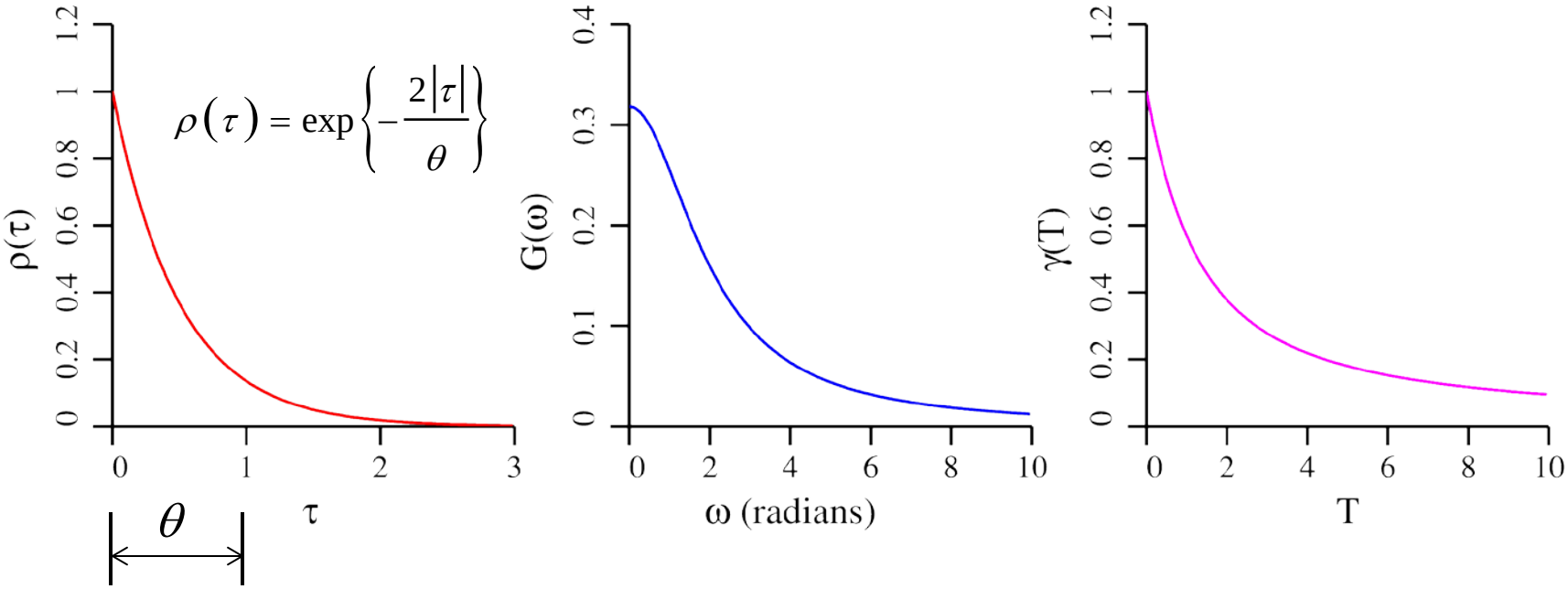
We expect that if two points are close together, they will be highly correlated – that is, if one is large, the other is also likely to be large.

We also expect that if two points are widely separated, they will tend to be uncorrelated – whether one is large or not has no (linear) influence on the size of the other.

We can capture these ideas with a correlation function, $\rho(\boldsymbol{x}, \boldsymbol{x}')$, which gives the correlation coefficient between $X(\boldsymbol{x})$ and $X(\boldsymbol{x}')$ and which decays as $\boldsymbol{\tau} = \boldsymbol{x} - \boldsymbol{x}'$ increases.

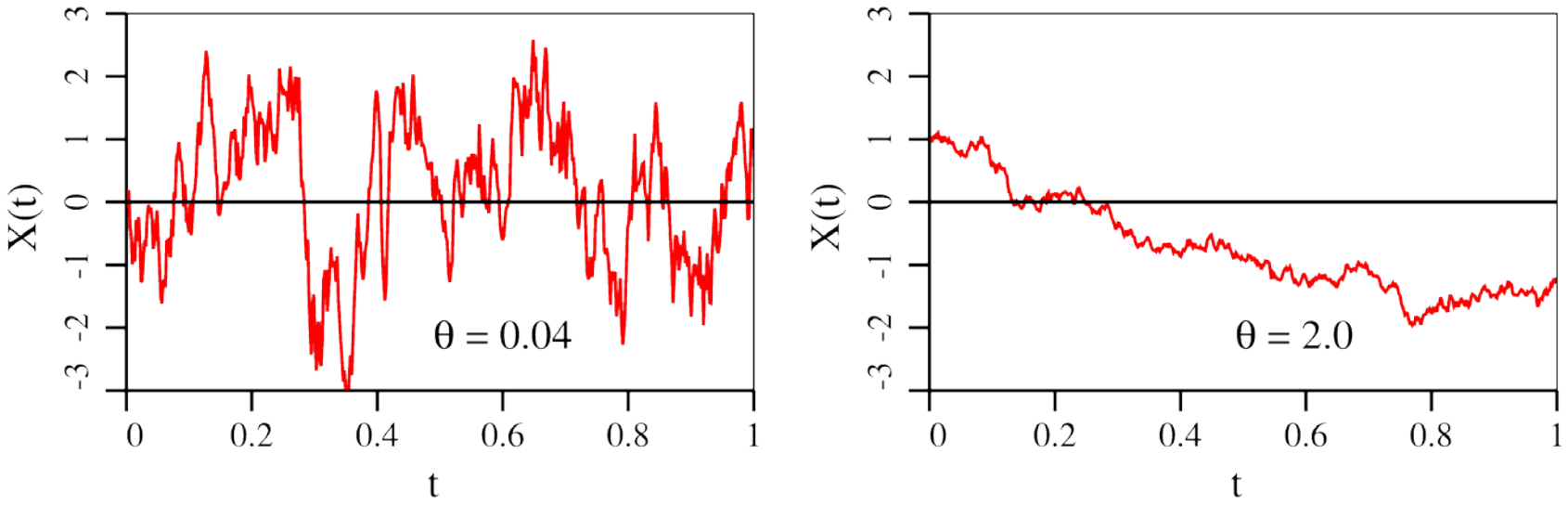
Correlation Structure

A common correlation function is $\rho(\mathbf{x},\mathbf{x}') = \exp\left(-\frac{2|\mathbf{x}-\mathbf{x}'|}{\theta}\right)$



θ is called the *correlation length* or *scale of fluctuation*. Roughly speaking, it measures the separation distance beyond which $X(\mathbf{x})$ and $X(\mathbf{x}')$ are effectively uncorrelated.

Effect of Correlation Length



- large correlation lengths have long wavelength components
- realizations appear to have a trend when viewed at scales less than the correlation length (both plots above are of *stationary* random processes).

Effect of Correlation Length

- when the correlation length is small, random fields tend to be rough,
- as $\theta \rightarrow 0$, all points in the field become uncorrelated with one another and the field becomes infinitely rough. This is physically unrealizable (we call this *white noise*),
- as the correlation length increases, the field becomes smoother.
- as $\theta \rightarrow \infty$, all points in the field become completely correlated (for *finite-scale* correlation functions). If the field is stationary, this means that the random field becomes completely uniform – different from realization to realization, but each realization is composed of a single random variable (traditional soil model).

The Effect of Averaging

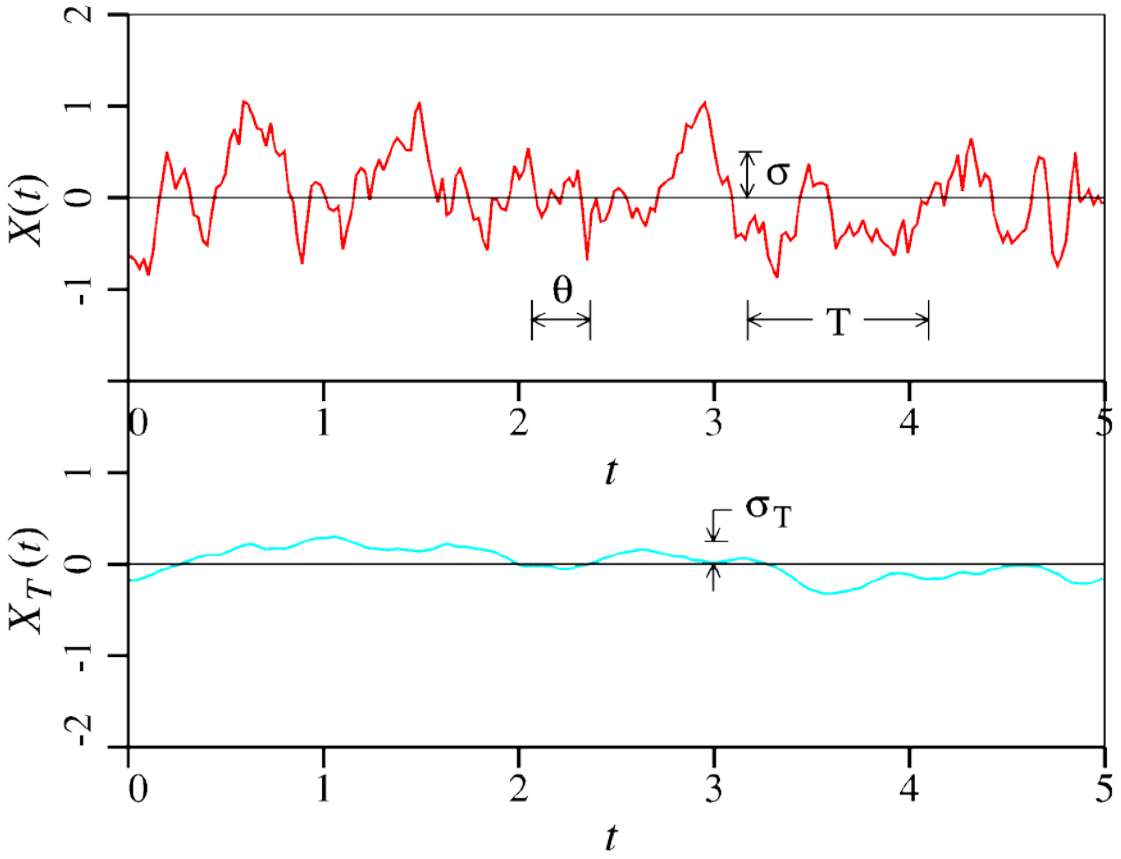


Illustration of the effect of local averaging on variance.

T is the moving window width over which the top plot is averaged to get the lower plot.

The Variance Function

Yet another 2nd moment characterization of random fields is the *variance function*.

The variance function measures the variance reduction due to averaging.

Consider the average $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

It is well known that $\sigma_{\bar{X}}^2 = \mathrm{Var}\left[\bar{X}\right] = \frac{\sigma_X^2}{n}$

if the X_i's are uncorrelated.

The variance function in this case is $\gamma(n) = \frac{1}{n}$

so that $\boxed{\sigma_{\bar{X}}^2 = \gamma(n)\sigma_X^2}$

The Variance Function

What happens if the X_i's are completely correlated?
If stationary and completely correlated, then

$$X_1 = X_2 = \cdots = X_n = X$$

now $$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{nX}{n} = X$$

so that $$\sigma_{\bar{X}}^2 = \sigma_X^2 \quad \Rightarrow \quad \gamma(n) = 1$$

In general: $$\frac{1}{n} \le \gamma(n) \le 1$$

Mathematically: $$\gamma(T) = \frac{1}{T^2}\int_0^T \int_0^T \rho(t-s)\,\mathrm{d}t\,\mathrm{d}s$$

$$= \text{average correlation}$$

The Variance Function