An introduction to numerical modelling of coupled problems in geomechanics

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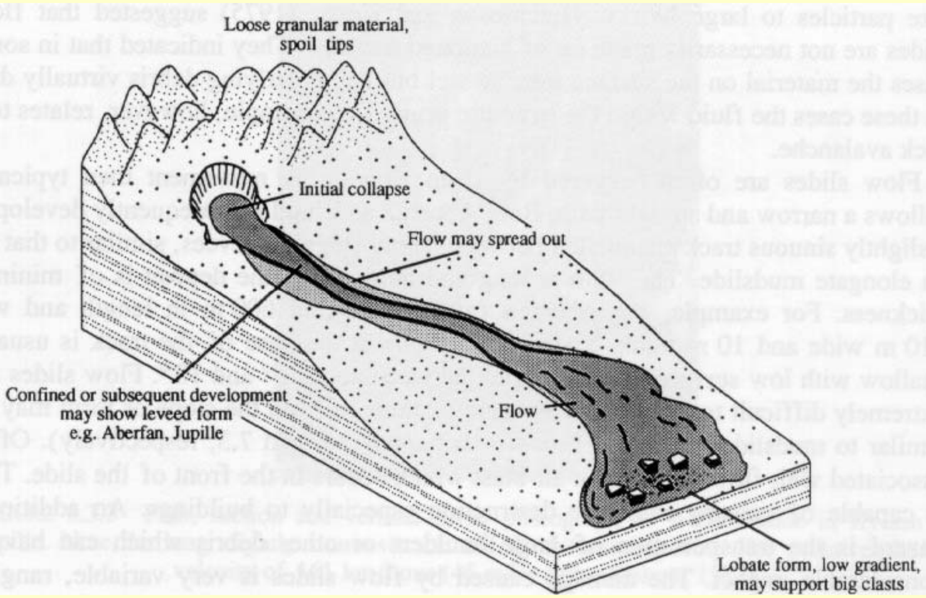
ETS de Ingenieros de Caminos
Madrid

Contents

Avalanches: Valtellina (July 1987)

28 th July 1987
40106 m^3
1550 - 2350 m

Flow slides



(Aberfan, 1966)

Flow slides (unsaturated soils)

Las Colinas (Sta Tecla)

Abi Barik village in Badakhshan http://www.theatlantic.com/

Debris Flows

Prof.Sucheng Zhang

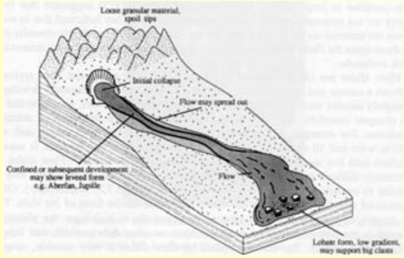
(Chengdu's Institute for Natural Hazards in Mountain Areas, CAS)


Mudflows:
Yunnan 2004

Problems to be solved
Which model?

- Granular avalanches

  Single phase



- Flowslides

  v-pw

- Debris flows, lahars

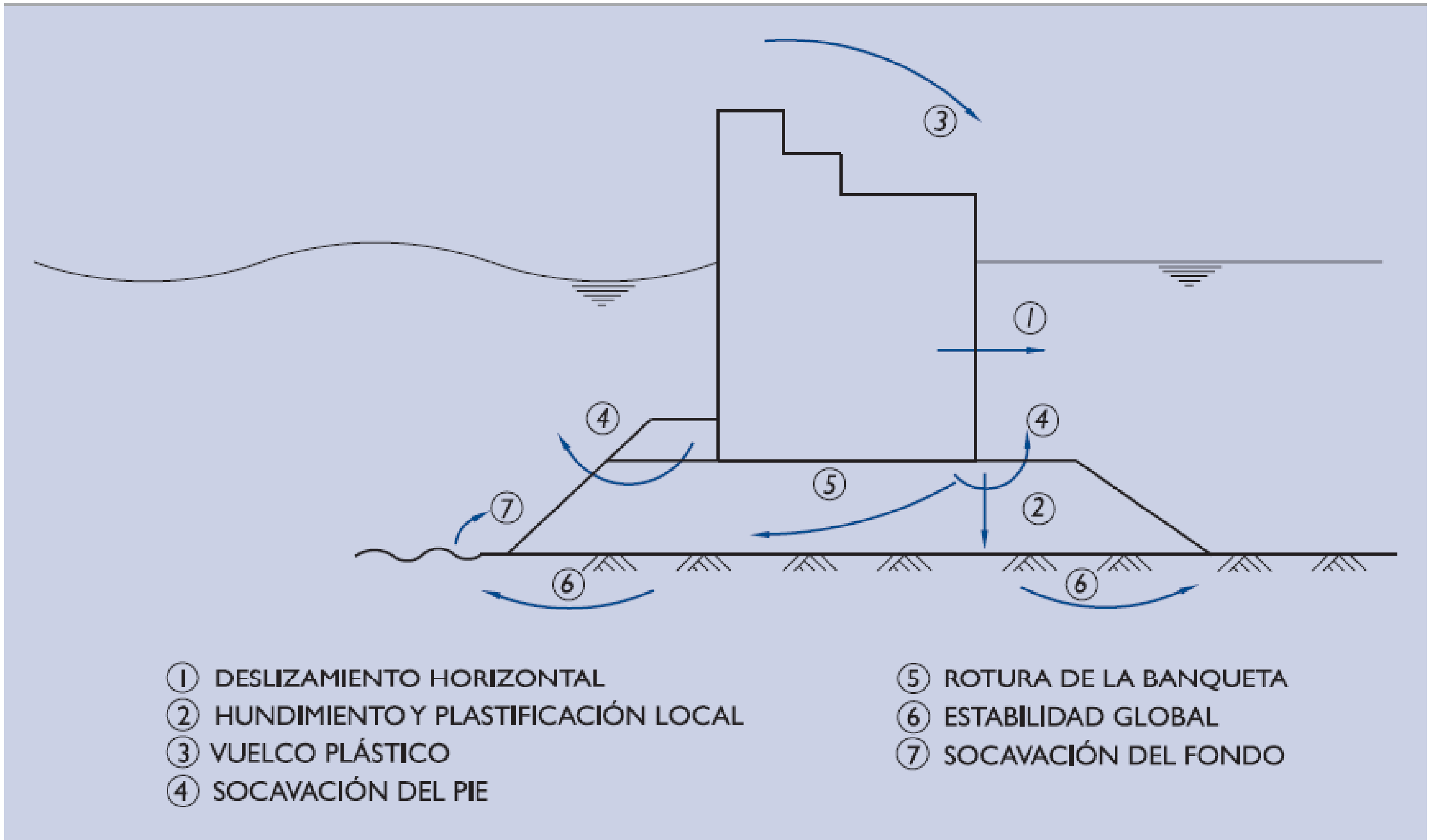
  vs-vw-pw

- Mudflows

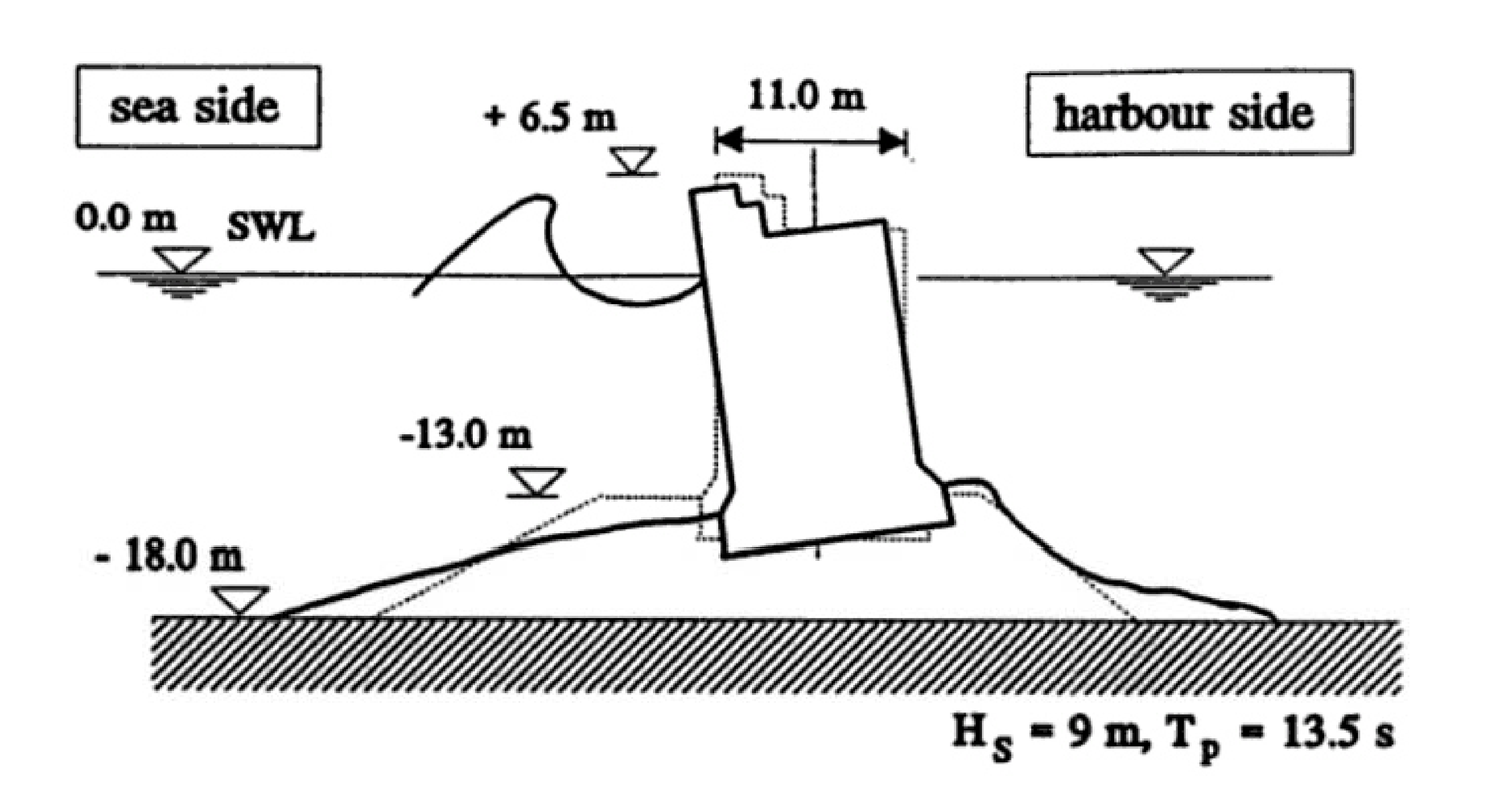
  Single phase

Foundations of marine structures

Figura 4.7.10. Modos de fallo geotécnicos adscritos a Estados Límite Últimos, asociados a los diques de abrigo verticales

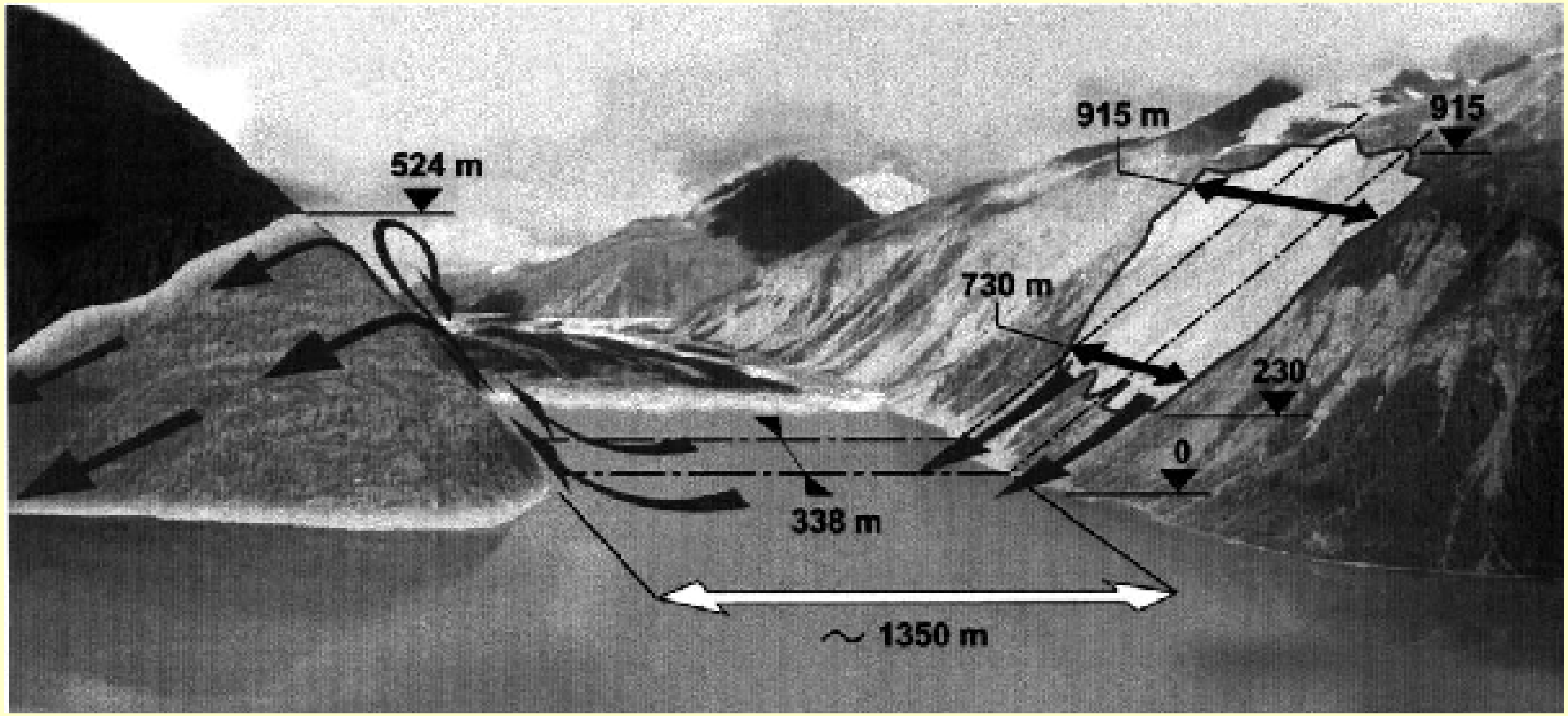


Mustapha's breakwater, Algier (1934)



Vaiont, 1963   2043 victims

"Megatsunamis": waves induced by landslides



Río Grijalba   Mexico (2008)


Río Grijalba
Mexico (2008)

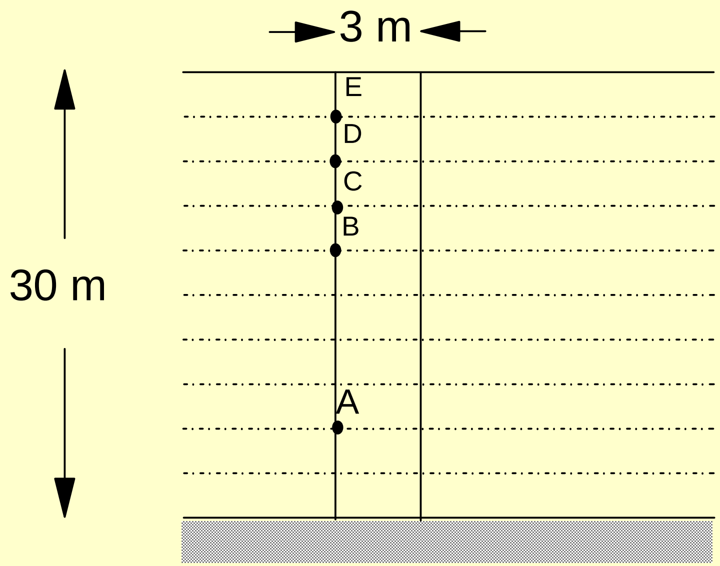
Complex phenomena: Lake Sarez (Tayikistán)

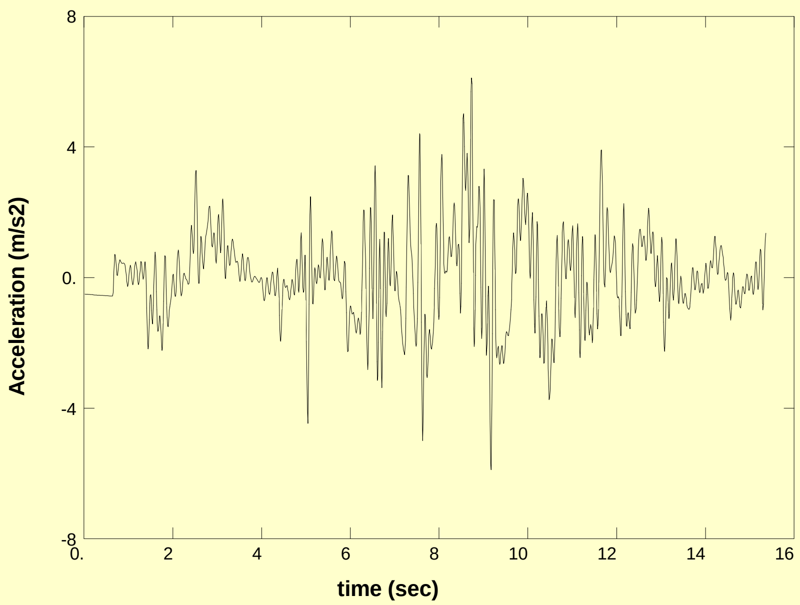

Más...
Mapa
Satélite
Relieve
Kudara
Barchidev
Sareskoe
A
5 km
2 mi
©2008 Google - Datos de mapa ©2008 AND, Europa Technologies -

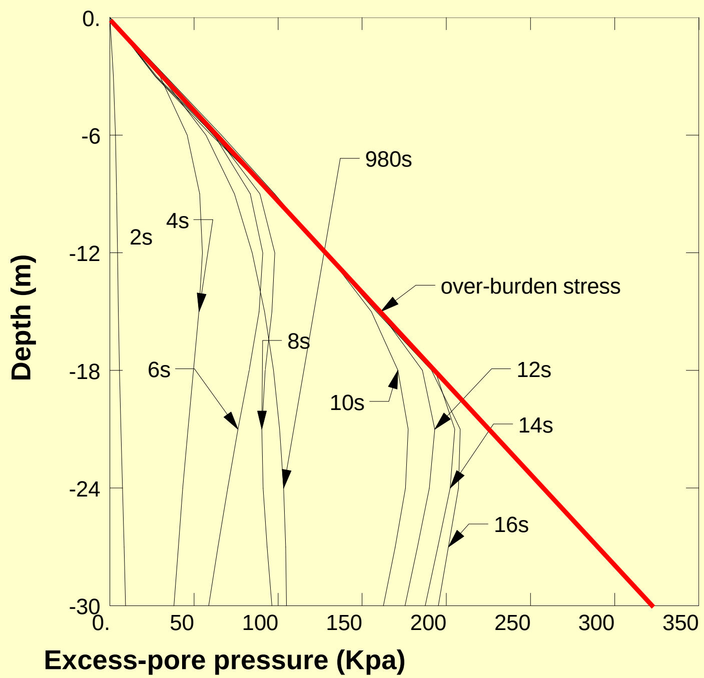

Más...
Mapa
Satélite
Relieve
1 km
1 mi
©2008 Google - Datos de mapa ©2008 AND, Europa Technologies - Términos de uso


Sichuan province, China
(2004)

Liquefaction of foundations and geostructures







Contents

- Introduction
- Mathematical Modelling

  Vs Vw pw

  u pw

  sigma v pw

  depth integrated
- Numerical Modelling (I) Finite elements

  FE: classic formulation

  Sigma v pw

  eulerian

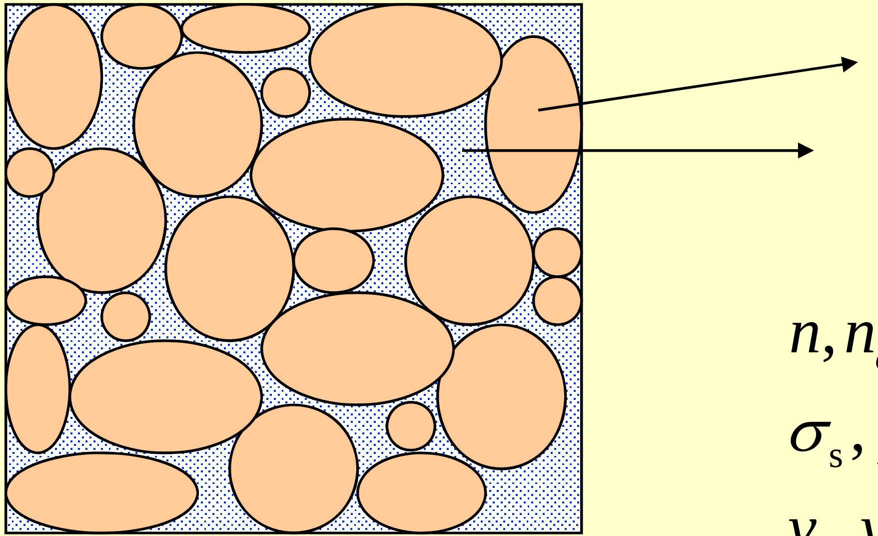
  depth integrated
- Numerical Modelling (II) SPH

  Basic formulation

  Large deformation. Coupling

  depth integrated

General Model: Unknowns



n, n_a, n_w volume fractions

σ_s, p_w, p_a stresses

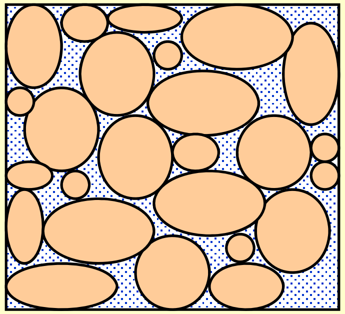
v_s, v_w, v_a velocities

d_s, d_w, d_a rate of deformation

Equations

- Balance of mass (water, air, mixture)
- Balance of momentum
- Constitutive or rheological
- Relations velocities – rate of deformations

Coupled model for saturated geomaterials (OCZ+Shiomi)



Balance of mass

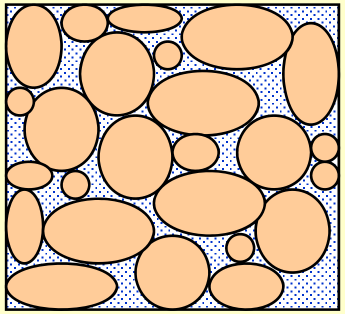
$$\frac{d^{(s)}}{dt}\left(\left(1-n\right)\rho_s\right)+\left(1-n\right)\rho_s \operatorname{div} v_s = 0$$

$$\frac{d^{(w)}}{dt}\left(n\rho_w\right)+n\rho_w \operatorname{div} v_w = 0$$

$$\frac{\left(1-n\right)}{K_s}\frac{d^{(s)}p_w}{dt}+\frac{n}{K_w}\frac{d^{(w)}p_w}{dt}+\operatorname{div} v_s+\operatorname{div} w = 0$$

$\operatorname{div} v_s + \operatorname{div} w = 0$ Incompressible grains and water

- Coupled model for saturated geomaterials (OCZ+Shiomi)



Balance of momentum

$$\left(1-n\right)\rho_s \frac{d^{(s)}v_s}{dt} = \mathrm{div}\,\sigma' - \left(1-n\right)\mathrm{grad}\,p_w + \left(1-n\right)\rho_s b + \left(1-n\right)R_s$$

$$n\rho_w \frac{d^{(w)}v_w}{dt} = -n\,\mathrm{grad}\,p_w + n\rho_w b + nR_w$$

$$nR_w = -\left(1-n\right)R_s = -R$$

$$R = C.\left(v_w - v_s\right)$$

$R = n^2 k_w^{-1}\left(v_w - v_s\right) = nk_w^{-1} w$ Darcy

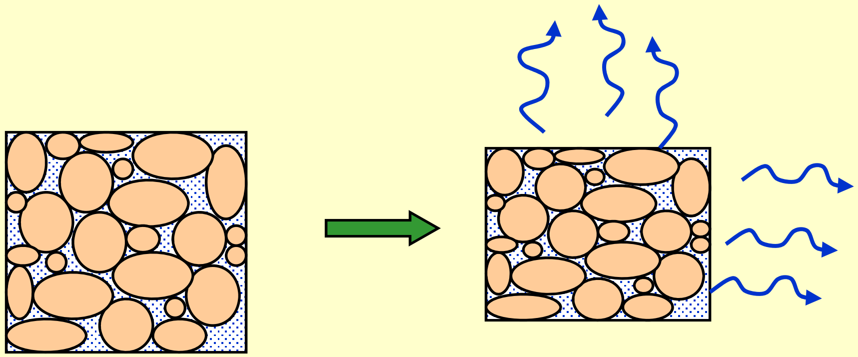
$R = \dfrac{n\left(1-n\right)}{V_T n^m}\left(\rho_s - \rho_w\right) g \left(v_w - v_s\right)$ Anderson

Classical Approach: Soils   What are Biot Equations?

- Balance of momentum (mixture)
- Balance of momentum (pore fluid: water air,...)
- Balance of mass (fluid)
- Constitutive
- Kinematic relations strain-displacement

Biot, Mandel, Zienkiewicz,
de Boer, Ehlers, Schrefler, Coussy,...

Biot-Zienkiewicz equations



$$v_w = v_s + v_{w/s}$$

$$v_{w/s} = \frac{w}{n} \quad \text{(Darcy)}$$

- Velocities of pore fluids relative to solid skeleton (small)
- Skeleton (lagrangian)
- Pore fluids (eulerian relative to skeleton)

Unknowns (Saturated)

v_s , w

σ, p_w

ε

Equations :

B1 Balance of momentum (mixture)
B2 Balance of mass (fluid)
B3 Balance of momentum (fluid)
B4 Constitutive (solid and fluid)
B5 Kinematics

Biot-Zienkiewicz equations: u-pw model

B1 $$\rho \frac{dv_s}{dt} = \rho b + div\left(\sigma' - \overline{p} I\right)$$

B2+B3 $$\frac{1}{Q^*}\frac{\partial p_w}{\partial t} = -\mathrm{div}\, v + \boxed{d_{v0}} + \mathrm{div}\left(k_w \,\mathrm{grad}\, p_w\right)$$

d_{v0} Extra dilatancy

B4 $$d\sigma' = D'.d\varepsilon$$

B5 $$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

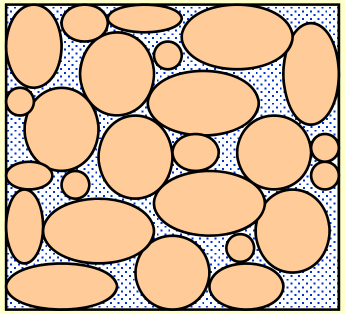
Unknowns :

v_s

σ, p_w

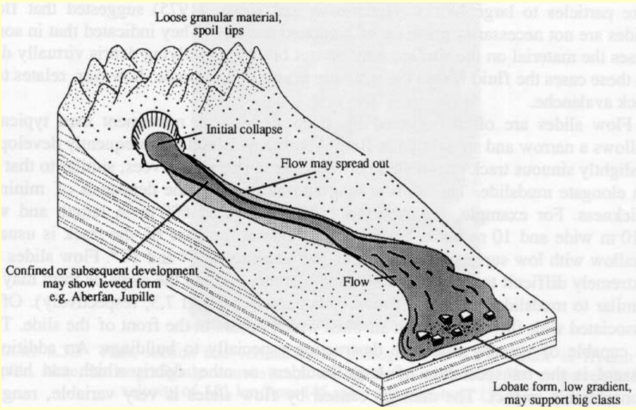
ε

Coupled model for saturated geomaterials (v-p_w)



Balance of momentum

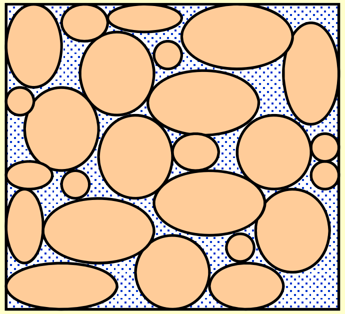
$$\frac{d^{(w)}}{dt} \approx \frac{d^{(s)}}{dt} = \frac{d}{dt} \qquad \frac{1}{Q} = \frac{(1-n)}{K_s} + \frac{n}{K_w}$$



$$\rho \frac{dv}{dt} = \operatorname{div} \sigma' - \operatorname{grad} p_w + \rho b$$

$$\frac{1}{Q}\frac{dp_w}{dt} + \operatorname{div} v_s - \operatorname{div}\left(k_w \operatorname{grad} p_w \right) = 0$$

Single phase model



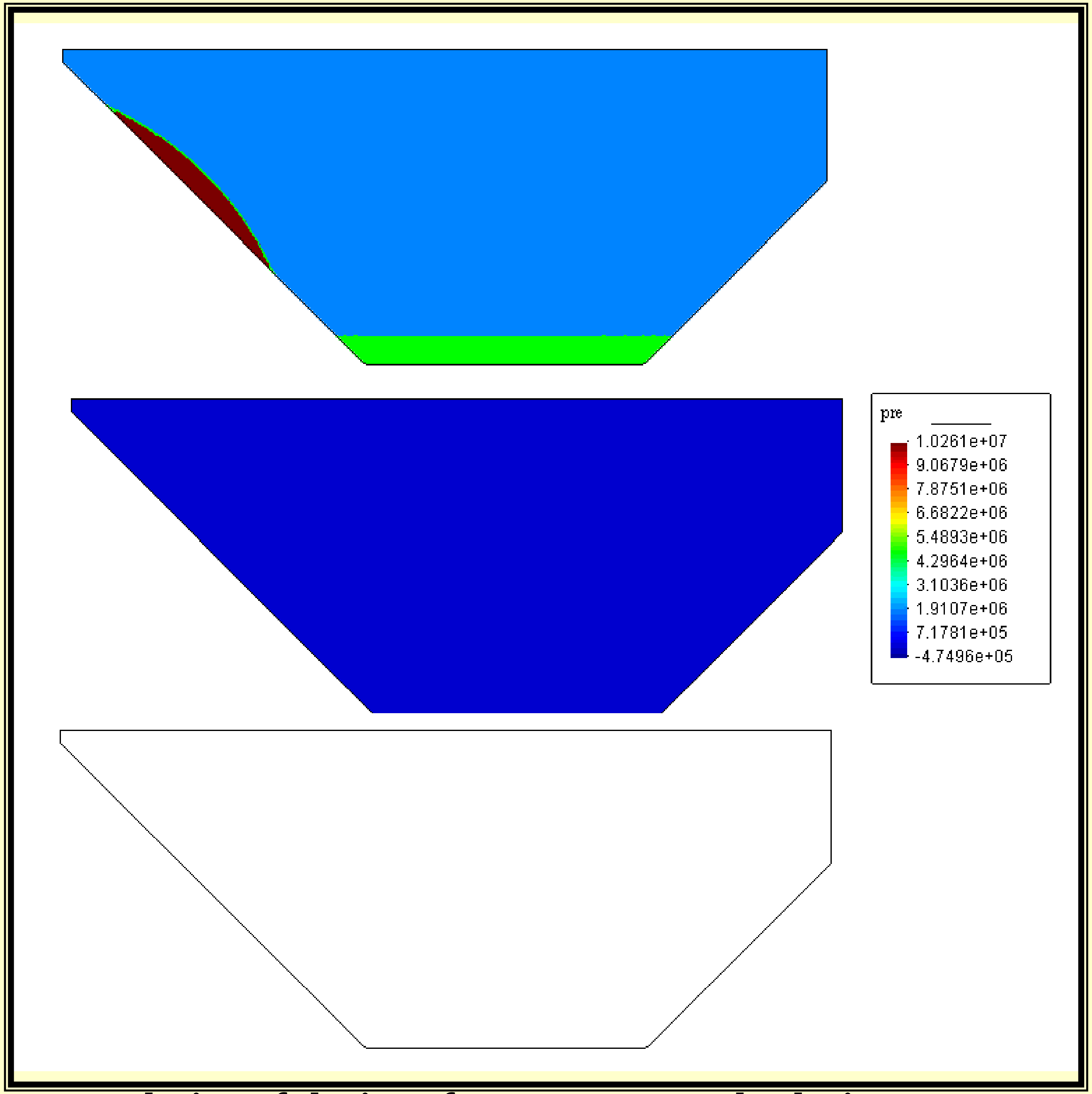
Equations

$$\frac{d\rho}{dt} + \rho \operatorname{div} v = 0$$

$$\rho \frac{dv}{dt} = \operatorname{div} \sigma + \rho b$$

$$\sigma = \sigma(d) \qquad \frac{d\sigma}{dt} = D : \frac{d\varepsilon}{dt} = D : d$$

$$d_{ij} = \frac{d\varepsilon_{ij}}{dt} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

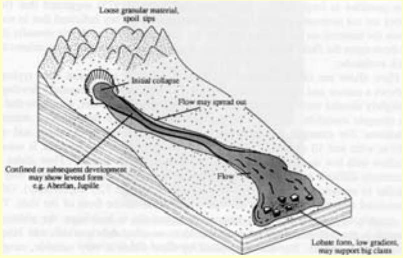


Evolution of the interfase, pressure and velocity vectors

Problems to be solved
Which model?

- Granular avalanches

  Single phase



- Flowslides

  v-pw

- Debris flows, lahars

  vs-vw-pw

- Mudflows

  Single phase

Alternative Approaches

- Classical

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

- 1st Order equations

$$\rho \frac{\partial v}{\partial t} = \frac{\partial \sigma}{\partial x}$$

$$\frac{\partial \sigma}{\partial t} = E \frac{\partial v}{\partial x}$$

$$\frac{\partial}{\partial t}\begin{pmatrix} \sigma \\ v \end{pmatrix} + \begin{pmatrix} 0 & -E \\ -1/\rho & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} \sigma \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial t} + \frac{\partial F}{\partial x} = 0$$

Viscoplasticity

$$\frac{\partial\sigma}{\partial t} = E\left(\dot{\varepsilon}^{e}\right) = E\left(\dot{\varepsilon} - \dot{\varepsilon}^{vp}\right)$$

$$\frac{\partial\sigma}{\partial t} = E\,\frac{\partial v}{\partial x} - E\dot{\varepsilon}^{vp} \qquad\qquad \rho\frac{\partial v}{\partial t} = \frac{\partial\sigma}{\partial x}$$

$$\frac{\partial}{\partial t}\begin{pmatrix}\sigma\\ v\end{pmatrix} + \begin{pmatrix}0 & -E\\ -\dfrac{1}{\rho} & 0\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}\sigma\\ v\end{pmatrix} = \begin{pmatrix}-E\dot{\varepsilon}^{vp}\\ 0\end{pmatrix}$$

$$\boxed{\frac{\partial}{\partial t}\begin{pmatrix}\sigma\\ v\end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix}-Ev\\ -\dfrac{v}{\rho}\end{pmatrix} = \begin{pmatrix}-E\dot{\varepsilon}^{vp}\\ 0\end{pmatrix} \quad \text{or} \quad \frac{\partial\phi}{\partial t} + \frac{\partial F}{\partial x} = s}$$

2D problems

Constitutive Eqn.

$$\frac{\partial}{\partial t}\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{pmatrix}=\begin{pmatrix}D_{11} & D_{12} & 0\\ D_{21} & D_{22} & 0\\ 0 & 0 & D_{33}\end{pmatrix}\cdot\begin{pmatrix}\dfrac{\partial v_1}{\partial x}\\ \dfrac{\partial v_2}{\partial y}\\ \dfrac{\partial v_1}{\partial y}+\dfrac{\partial v_2}{\partial x}\end{pmatrix}$$

Balance of Momentum

$$\rho\frac{\partial}{\partial t}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\frac{\partial}{\partial x}\begin{pmatrix}\sigma_{11}\\ \sigma_{12}\end{pmatrix}+\frac{\partial}{\partial y}\begin{pmatrix}\sigma_{12}\\ \sigma_{22}\end{pmatrix}$$

$$\Rightarrow\quad \frac{\partial}{\partial t}\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\\ v_1\\ v_2\end{pmatrix}-\frac{\partial}{\partial x}\begin{pmatrix}D_{11}v_1\\ D_{12}v_2\\ D_{33}v_2\\ \sigma_{11}/\rho\\ \sigma_{12}/\rho\end{pmatrix}-\frac{\partial}{\partial y}\begin{pmatrix}D_{12}v_2\\ D_{22}v_2\\ D_{33}v_1\\ \sigma_{12}/\rho\\ \sigma_{22}/\rho\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\\ 0\\ 0\end{pmatrix}$$

2D problems

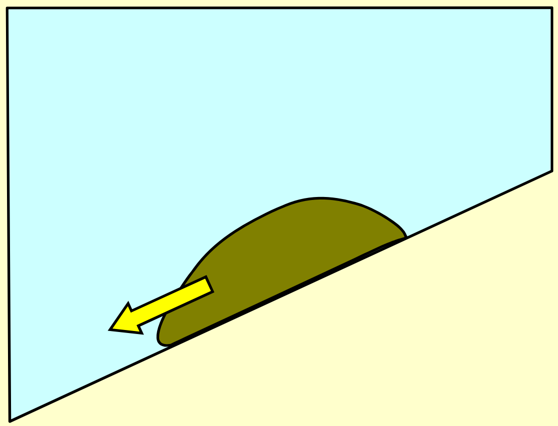
$$\frac{\partial}{\partial t}\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\\ v_1\\ v_2\end{pmatrix}-\frac{\partial}{\partial x}\begin{pmatrix}D_{11}v_1\\ D_{12}v_2\\ D_{33}v_2\\ \sigma_{11}/\rho\\ \sigma_{12}/\rho\end{pmatrix}-\frac{\partial}{\partial y}\begin{pmatrix}D_{12}v_2\\ D_{22}v_2\\ D_{33}v_1\\ \sigma_{12}/\rho\\ \sigma_{22}/\rho\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\\ 0\\ 0\end{pmatrix}$$

$$\frac{\partial\phi}{\partial t}+\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}=0$$

or $$\frac{\partial\phi}{\partial t}+\operatorname{div}F=0$$

Viscoplasticity $\Longrightarrow$ $$\frac{\partial\phi}{\partial t}+\operatorname{div}F=s$$

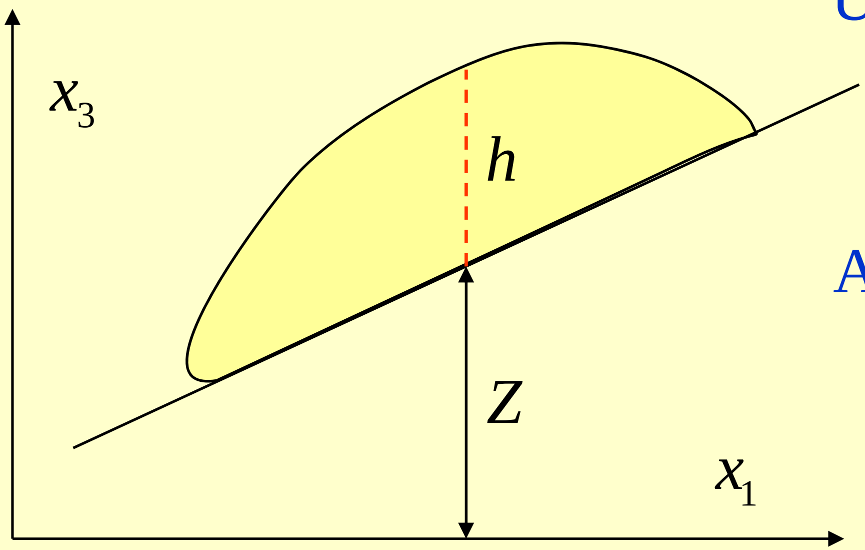
Depth Integrated Models I. Single phase



Unknowns: $v_0, (\sigma, d)$ and p_w

Problems:

- Interfaces (or free surface)



Unknowns:
$$\begin{cases} h \\ h\,\overline{v}_1 = \int v_1 dx_3 \\ h\,\overline{p}_w = \int p_w dx_3 \end{cases}$$

Advantages:

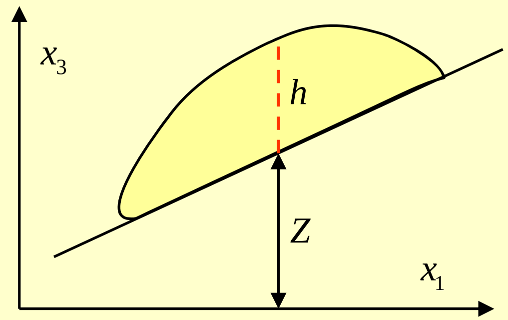
- No interfaces
- Less unknowns (1 dim less)

Depth Integrated Models I. Single phase

- From 3D Models…

$$\rho_m \frac{Dv_0}{Dt} = div\sigma + \rho_m b$$

$$div\, v_0 = 0 \qquad \text{(Incompressible)}$$



- Use Leibnitz rule

$$\int_a^b \frac{\partial}{\partial s} F(r,s)\,dr = \frac{\partial}{\partial s}\int_a^b F(r,s)\,dr - F(b,s)\frac{\partial b}{\partial s} + F(a,s)\frac{\partial a}{\partial s}$$

- Apply it to **balance of mass** equation $div\, v_0 = 0$

$$\int_Z^{Z+h} \frac{\partial v_i}{\partial x_i} dx_3 = \frac{\partial}{\partial x_i}\int_Z^{Z+h} v_i dx_3 - v_i \left.\frac{\partial (Z+h)}{\partial x_i}\right|_{Z+h} + v_i \left.\frac{\partial (Z)}{\partial x_i}\right|_Z = 0$$

$i = 1,2$

$$\boxed{\frac{\partial h}{\partial t} + \frac{\partial}{\partial x_1}\left(h\bar{v}_1\right) + \frac{\partial}{\partial x_2}\left(h\bar{v}_2\right) = e_R}$$

$$\boxed{\frac{\bar{d}h}{dt} = h\frac{\partial \bar{v}_1}{\partial x_1} + h\frac{\partial \bar{v}_2}{\partial x_2} + e_R}$$

$$\begin{cases} h\bar{v}_1 = \int v_1 dx_3 \\ \dfrac{\bar{d}h}{dt} = \dfrac{\partial h}{\partial t} + \bar{v}_1\dfrac{\partial h}{\partial x_1} + \bar{v}_2\dfrac{\partial h}{\partial x_2} \\ e_R \quad \text{Erosion rate} \end{cases}$$

Eulerian

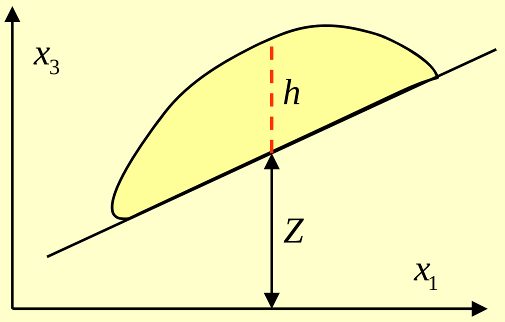
Pseudo- Lagrangian

Depth Integrated Models I. Single phase

- Balance of momentum

$$\rho_m \frac{Dv_0}{Dt} = div\sigma + \rho_m b$$

- Apply Leibnitz rule
- Quasi lagrangian form



$$\rho h \frac{dv}{dt} = \frac{1}{2}\rho \,\mathrm{grad}\left(h^2 b_3\right) + \rho h b_3 \,\mathrm{grad}\, Z + \tau_b - e_R \overline{v}$$

b_3 body forces

ρ mixture density

τ_b shear stress at bottom

e_R erosion coefficient $= E_s h \left|\overline{v}\right|$

$$E_s \approx \frac{\ln\left(V_{final} / V_0\right)}{\mathrm{distance}} \quad \text{(Hungr)}$$

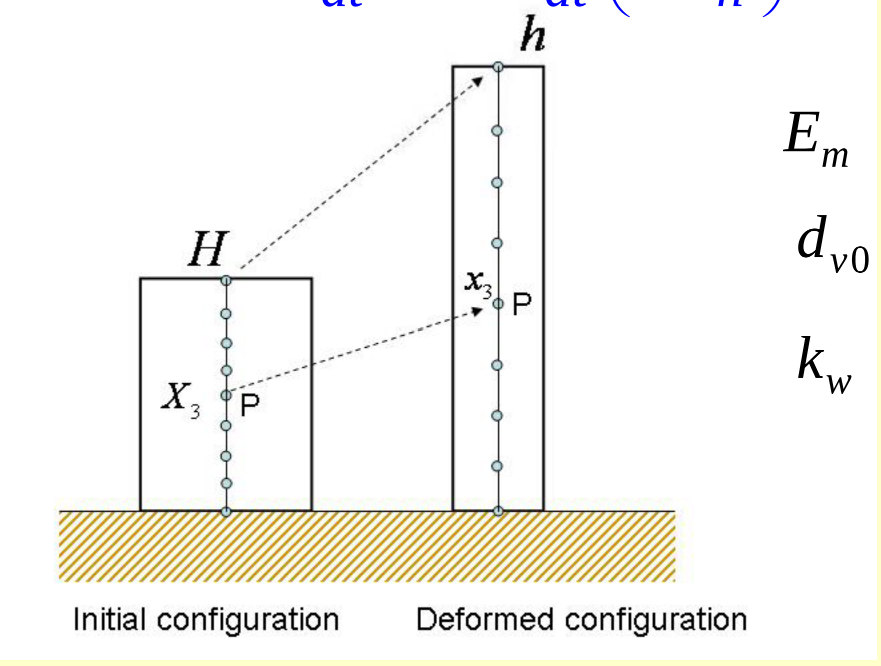
Depth Integrated Models II. v-p_w

$$\frac{\bar{d}h}{dt} + h\,\mathrm{div}\,\bar{v} = e_R$$

$$\rho h \frac{dv}{dt} = \frac{1}{2}\rho\,\mathrm{grad}\left(h^2 b_3\right) + \rho h b_3\,\mathrm{grad}\,Z + \tau_b - e_R \bar{v}$$

- 1D consolidation along depth

$$\frac{dp_w}{dt} = \rho b_3 \frac{dh}{dt}\left(1 - \frac{x_3}{h}\right) + E_m d_{v0} + k_w E_m \frac{\partial^2 p_w}{\partial {x_3}^2}$$



E_m Edometric modulus

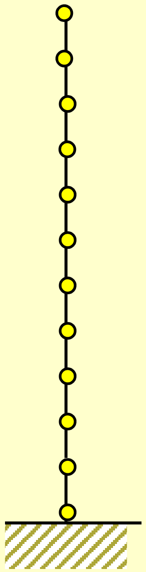
d_{v0} extra dilatancy

k_w permeability

- Depth Integrated Models II. v-pw (consolidation along depth)

$$\frac{dp_w}{dt} = \frac{d\sigma}{dt} + \frac{\partial}{\partial x_3}\left(\boxed{c_v \frac{\partial p_{w1}}{\partial x_3}} \right) + E_m d_{V01}$$

⇨ Use a FD explicit scheme



Notes:

- Depth changes:
  - Mesh changes too
  - Total stress and Pwp change

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  Vs Vw pw

  u pw

  sigma v pw

  depth integrated
- Numerical Modelling (I) Finite elements

  FE: classic formulation

  Sigma v pw

  eulerian

  depth integrated
- Numerical Modelling (II) SPH

  Basic formulation

  Large deformation. Coupling

  depth integrated

- Biot-Zienkiewicz equations: u-pw model

$$u = N_u \, \hat{u} \qquad \mathbf{p_w} = \mathbf{N_p} \, \hat{\mathbf{p}}_\mathbf{w}$$

$$M \frac{d^2 \bar{u}}{dt^2} + \int_\Omega B^T . \sigma' \, d\Omega - Q \bar{p}_w - f_u = 0$$

$$Q^T \frac{d\bar{u}}{dt} + H . \bar{p}_w + C . \frac{d\bar{p}_w}{dt} - f_p = 0$$

$$\mathbf{M} = \int_\Omega \rho \mathbf{N}_u^T \mathbf{N}_u d\Omega \qquad C = \int_\Omega \frac{1}{Q^*} \mathbf{N}_p^T \mathbf{N}_p d\Omega$$

$$Q = \int_\Omega S_w \alpha B^T m \mathbf{N}_p d\Omega \qquad H = \int \nabla \mathbf{N}_p^T k_w \nabla \mathbf{N}_p d\Omega$$

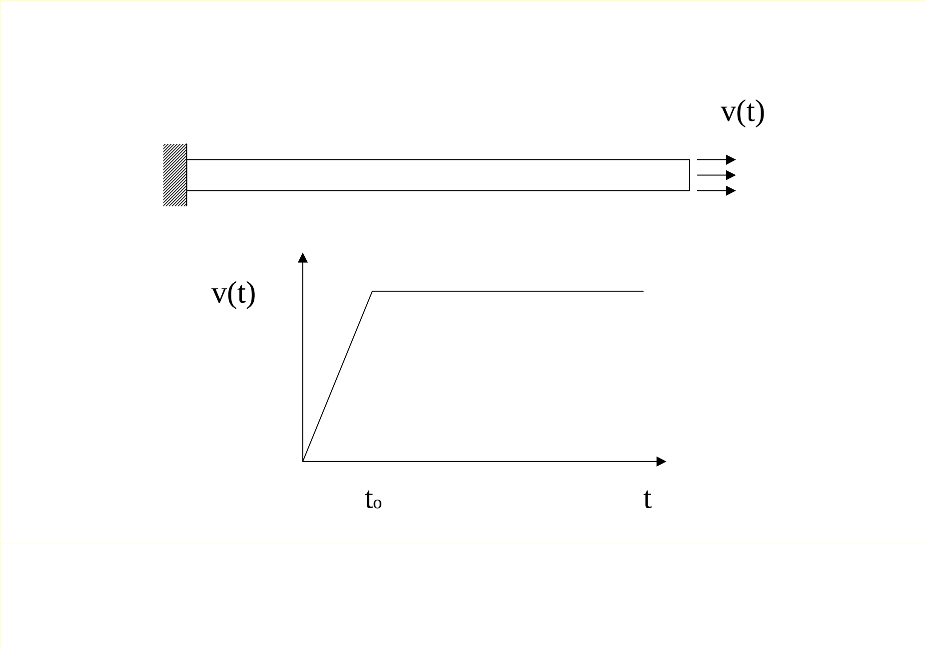
- Biot-Zienkiewicz equations: u-pw model

$$M\frac{d^2\bar{u}}{dt^2}+\int_\Omega B^T.\sigma' d\Omega - Q\bar{p}_w - f_u = 0$$

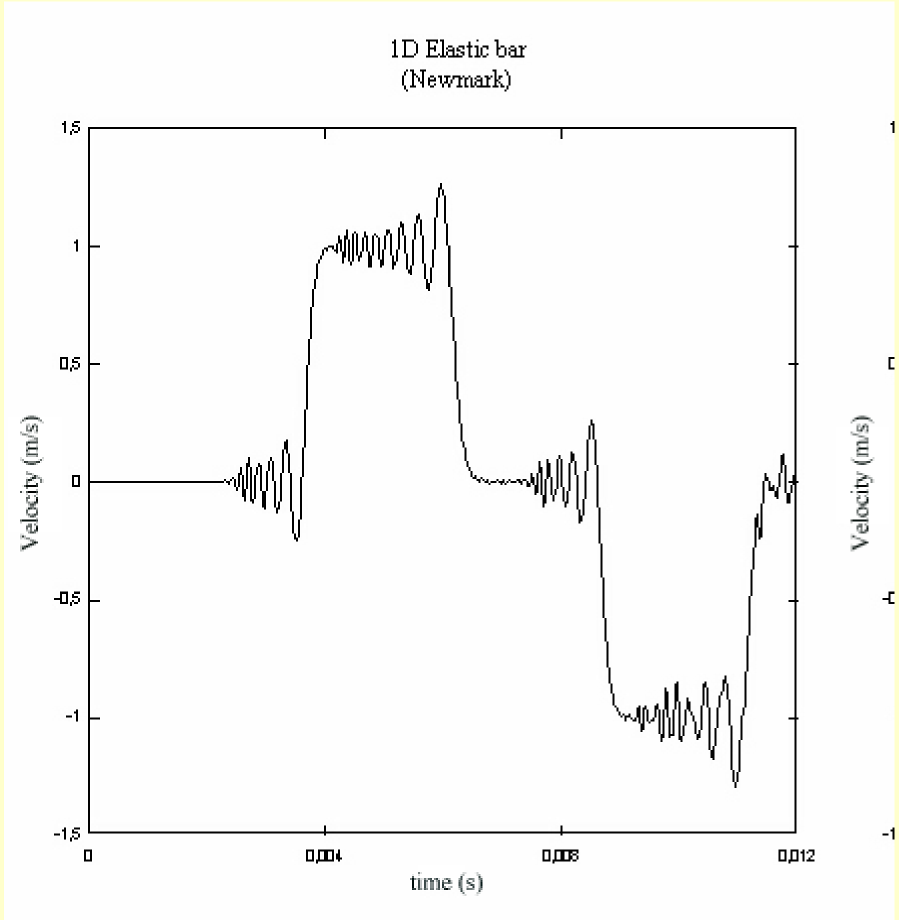
$$Q^T\frac{d\bar{u}}{dt}+H.\bar{p}_w + C.\frac{d\bar{p}_w}{dt} - f_p = 0$$

$$\boxed{\begin{aligned} &M\Delta\ddot{\bar{u}}^n + \int B^T\sigma'^{n+1} - \theta\Delta t Q\Delta\dot{\bar{p}}_w^n - F_u^{n+1} = \Phi_u = 0 \\ &\beta_1\Delta t Q^T\Delta\ddot{\bar{u}}^n + (\Delta t\theta H + C)\Delta\dot{\bar{p}}_w^n - F_p^{n+1} = \Phi_p = 0 \end{aligned}}$$

What can go wrong?

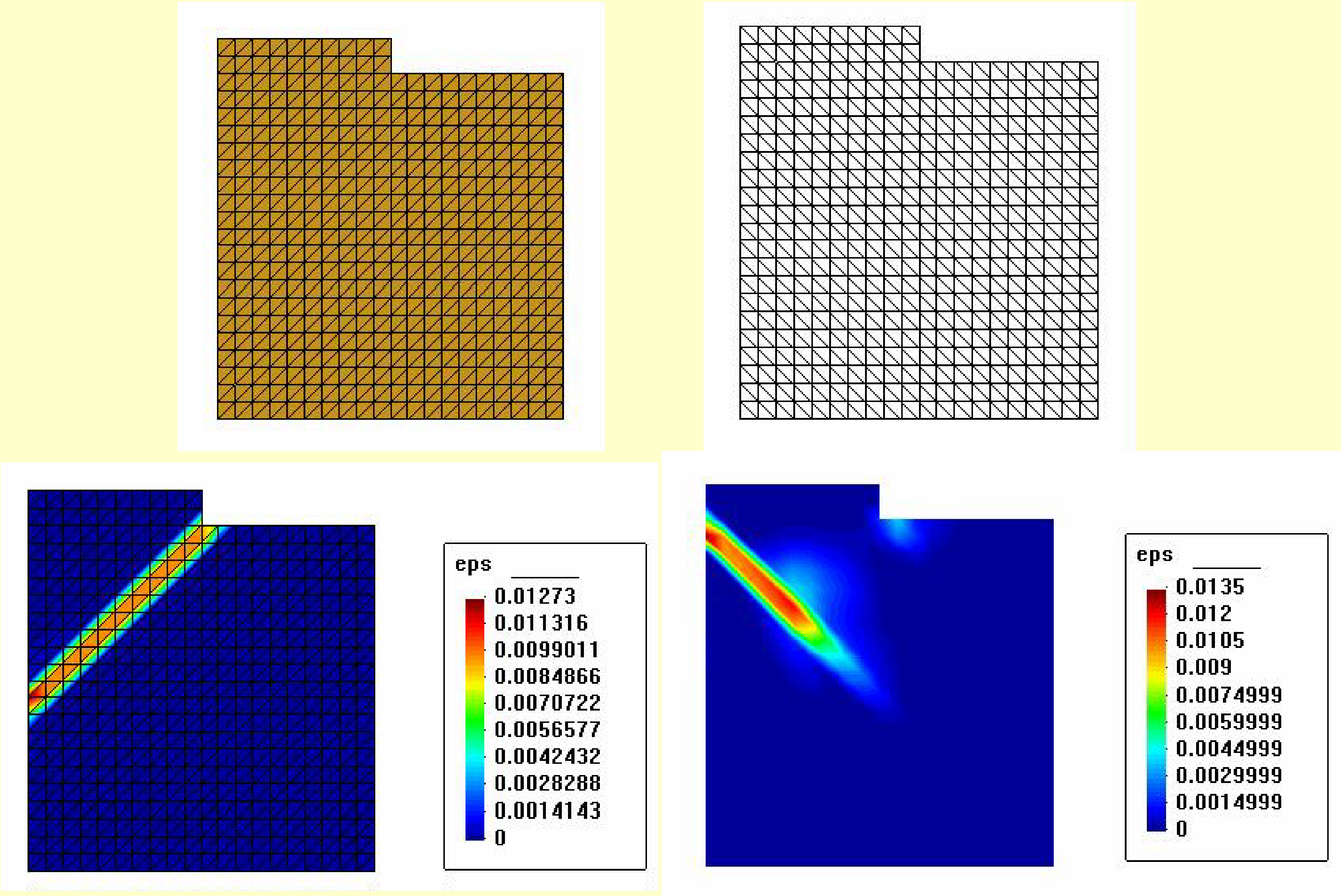


- Numerical damping
- Numerical dispersion



v at x = L/4

What can go wrong? (II)



Alternative Approaches

- Classical

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

- 1st Order equations

$$\rho \frac{\partial v}{\partial t} = \frac{\partial \sigma}{\partial x}$$

$$\frac{\partial \sigma}{\partial t} = E \frac{\partial v}{\partial x}$$

$$\frac{\partial}{\partial t}\begin{pmatrix}\sigma \\ v\end{pmatrix} + \begin{pmatrix}0 & -E \\ -1/\rho & 0\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}\sigma \\ v\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial t} + \frac{\partial F}{\partial x} = 0$$

Viscoplasticity

$$\frac{\partial \sigma}{\partial t} = E\left(\dot{\varepsilon}^{e}\right) = E\left(\dot{\varepsilon} - \dot{\varepsilon}^{vp}\right)$$

$$\frac{\partial \sigma}{\partial t} = E\frac{\partial v}{\partial x} - E\dot{\varepsilon}^{vp} \qquad \rho\frac{\partial v}{\partial t} = \frac{\partial \sigma}{\partial x}$$

$$\frac{\partial}{\partial t}\begin{pmatrix}\sigma \\ v\end{pmatrix} + \begin{pmatrix}0 & -E \\ -\dfrac{1}{\rho} & 0\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}\sigma \\ v\end{pmatrix} = \begin{pmatrix}-E\dot{\varepsilon}^{vp} \\ 0\end{pmatrix}$$

$$\boxed{\frac{\partial}{\partial t}\begin{pmatrix}\sigma \\ v\end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix}-Ev \\ -\dfrac{v}{\rho}\end{pmatrix} = \begin{pmatrix}-E\dot{\varepsilon}^{vp} \\ 0\end{pmatrix} \quad \text{or} \quad \frac{\partial \phi}{\partial t} + \frac{\partial F}{\partial x} = s}$$

2D problems

Constitutive Eqn.

$$\frac{\partial}{\partial t}\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{pmatrix}=\begin{pmatrix}D_{11} & D_{12} & 0\\ D_{21} & D_{22} & 0\\ 0 & 0 & D_{33}\end{pmatrix}.\begin{pmatrix}\dfrac{\partial v_1}{\partial x}\\ \dfrac{\partial v_2}{\partial y}\\ \dfrac{\partial v_1}{\partial y}+\dfrac{\partial v_2}{\partial x}\end{pmatrix}$$

Balance of Momentum

$$\rho\frac{\partial}{\partial t}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\frac{\partial}{\partial x}\begin{pmatrix}\sigma_{11}\\ \sigma_{12}\end{pmatrix}+\frac{\partial}{\partial y}\begin{pmatrix}\sigma_{12}\\ \sigma_{22}\end{pmatrix}$$

$$\Rightarrow\quad \frac{\partial}{\partial t}\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\\ v_1\\ v_2\end{pmatrix}-\frac{\partial}{\partial x}\begin{pmatrix}D_{11}v_1\\ D_{12}v_2\\ D_{33}v_2\\ \sigma_{11}/\rho\\ \sigma_{12}/\rho\end{pmatrix}-\frac{\partial}{\partial y}\begin{pmatrix}D_{12}v_2\\ D_{22}v_2\\ D_{33}v_1\\ \sigma_{12}/\rho\\ \sigma_{22}/\rho\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\\ 0\\ 0\end{pmatrix}$$

2D problems

$$\frac{\partial}{\partial t}\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\\ v_1\\ v_2\end{pmatrix}-\frac{\partial}{\partial x}\begin{pmatrix}D_{11}v_1\\ D_{12}v_2\\ D_{33}v_2\\ \sigma_{11}/\rho\\ \sigma_{12}/\rho\end{pmatrix}-\frac{\partial}{\partial y}\begin{pmatrix}D_{12}v_2\\ D_{22}v_2\\ D_{33}v_1\\ \sigma_{12}/\rho\\ \sigma_{22}/\rho\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}$$

$$\frac{\partial\phi}{\partial t}+\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}=0$$

or $$\frac{\partial\phi}{\partial t}+\operatorname{div}F=0$$

Viscoplasticity $\Longrightarrow$ $$\frac{\partial\phi}{\partial t}+\operatorname{div}F=s$$

- Discretization: 2 steps Taylor Galerkin

$$\frac{\partial \phi}{\partial t} + \frac{\partial F}{\partial x} = s \quad \Longrightarrow \quad \frac{\partial \phi}{\partial t} = s - \frac{\partial F}{\partial x}$$

$$\phi^{n+1} = \phi^n + \Delta t \left.\frac{\partial \phi}{\partial t}\right|^{n+\frac{1}{2}} \qquad \left.\frac{\partial \phi}{\partial t}\right|^{n+\frac{1}{2}} = s - \left.\frac{\partial F}{\partial x}\right|^{n+\frac{1}{2}}$$

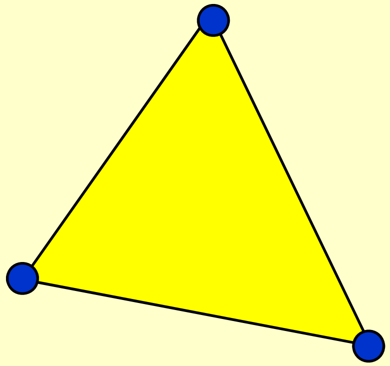
Step 1

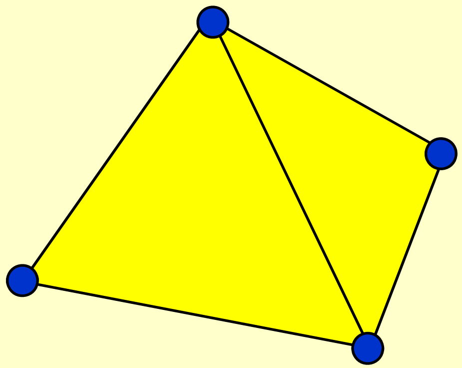
$$\phi^{n+\frac{1}{2}} = \phi^n + \frac{\Delta t}{2}\left.\frac{\partial \phi}{\partial t}\right|^{n} = \phi^n + \frac{\Delta t}{2}\left.\left(s - \frac{\partial F}{\partial x}\right)\right|^{n}$$

$$\phi^{n+\frac{1}{2}} \Longrightarrow F^{n+\frac{1}{2}} \quad s^{n+\frac{1}{2}}$$

Step 2

$$\phi^{n+1} = \phi^n + \Delta t \left(s - \frac{\partial F}{\partial x}\right)^{n+\frac{1}{2}}$$

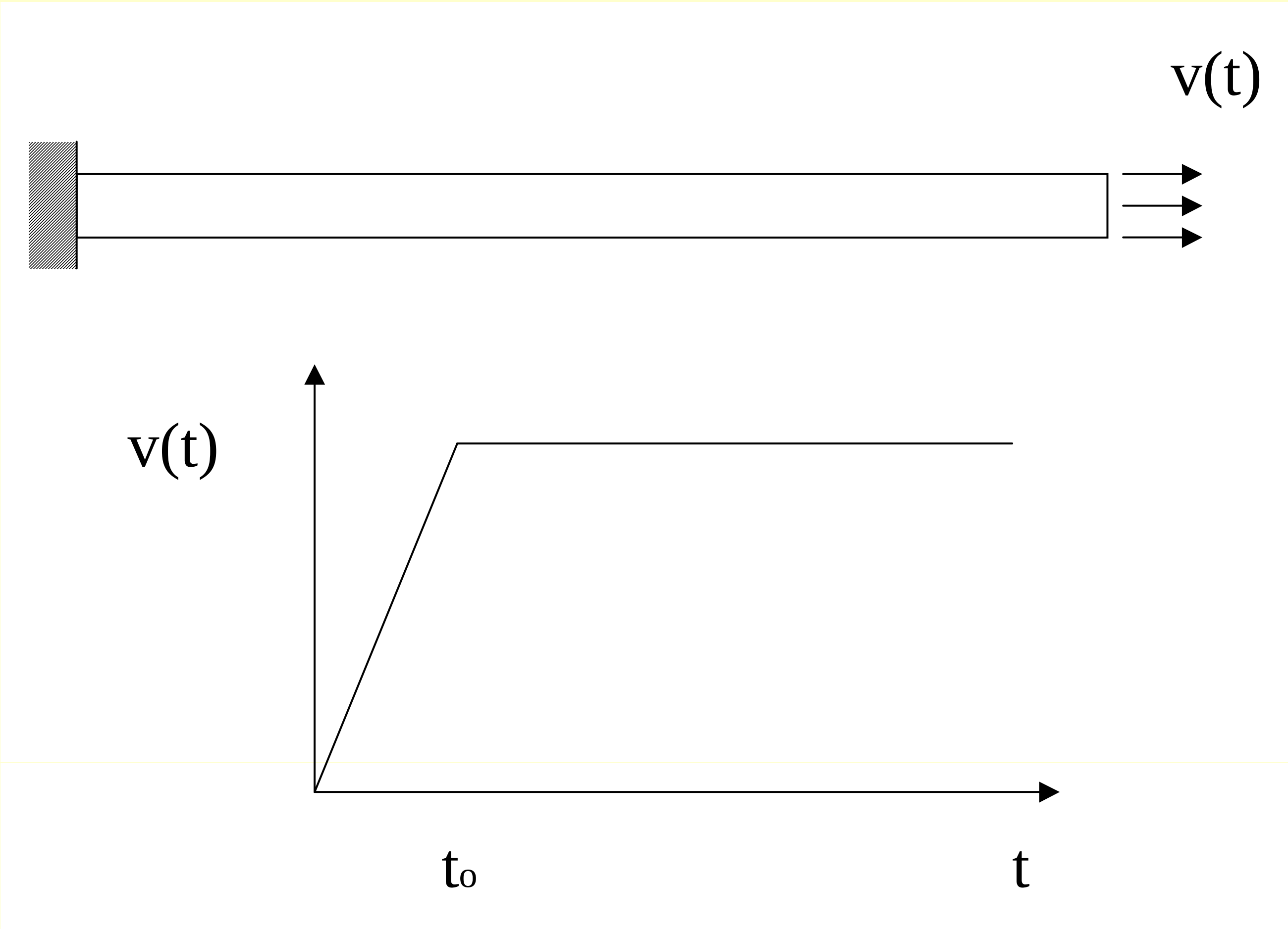


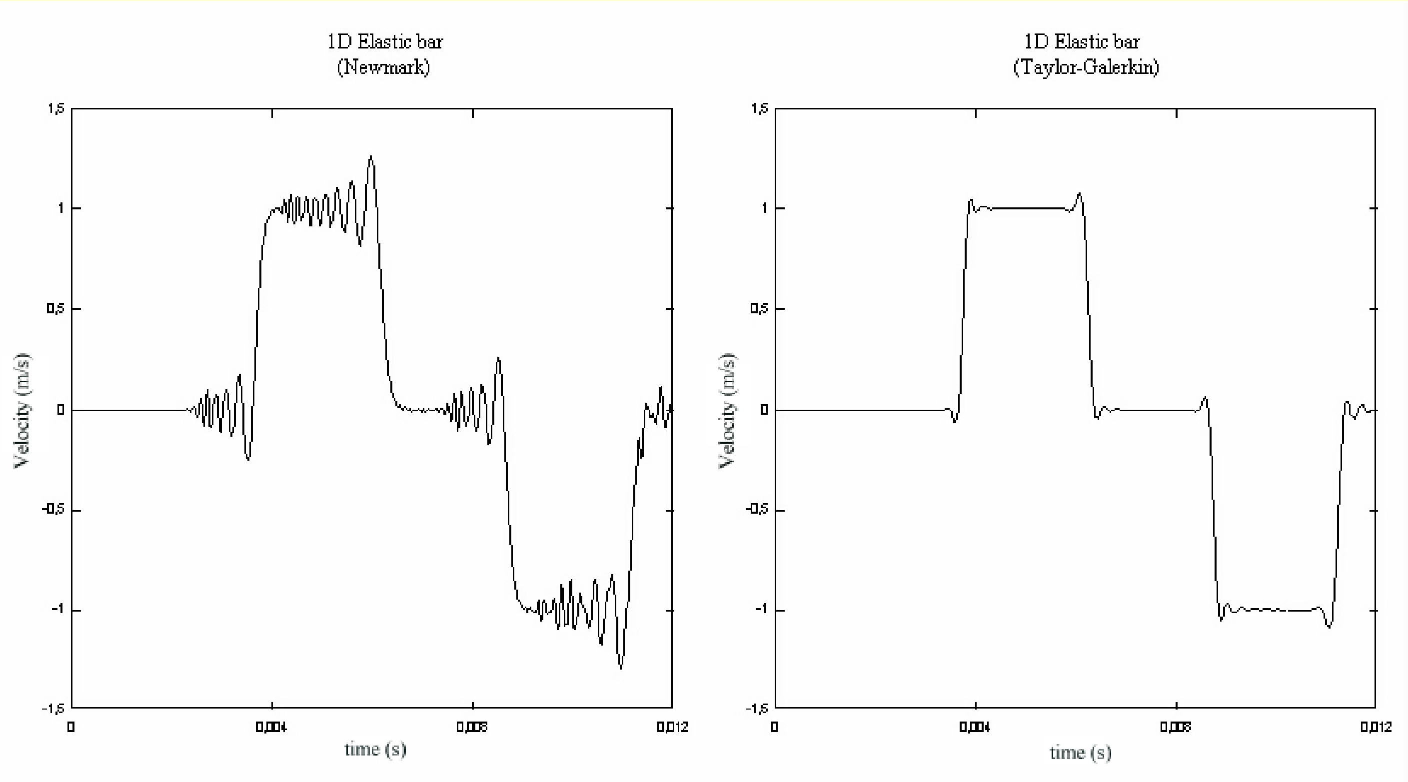


Linear triangles (2D) or tetrahedra (3D)

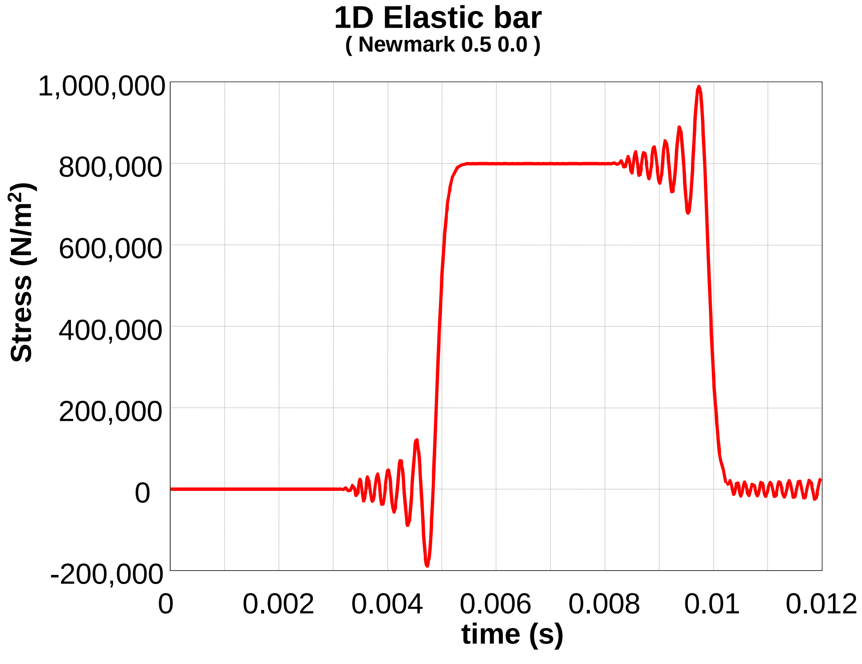
- Equal order of interpolation
- Faster codes
- Extremely good performance in bending
- Robusts in plasticity
- Require stabilization
  when material is incompressible
  (Babuska-Brezzi conditions)

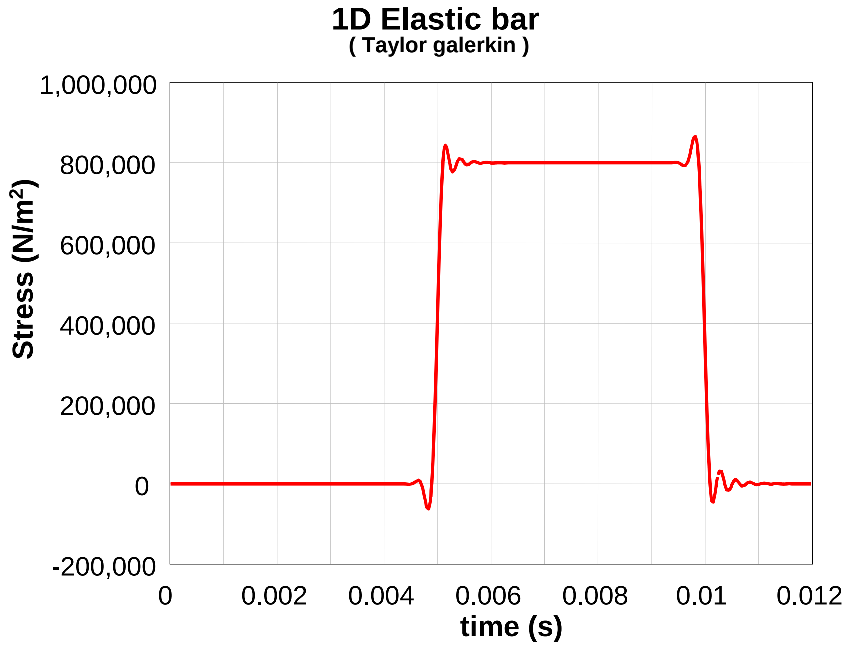
Example: 1D Bar



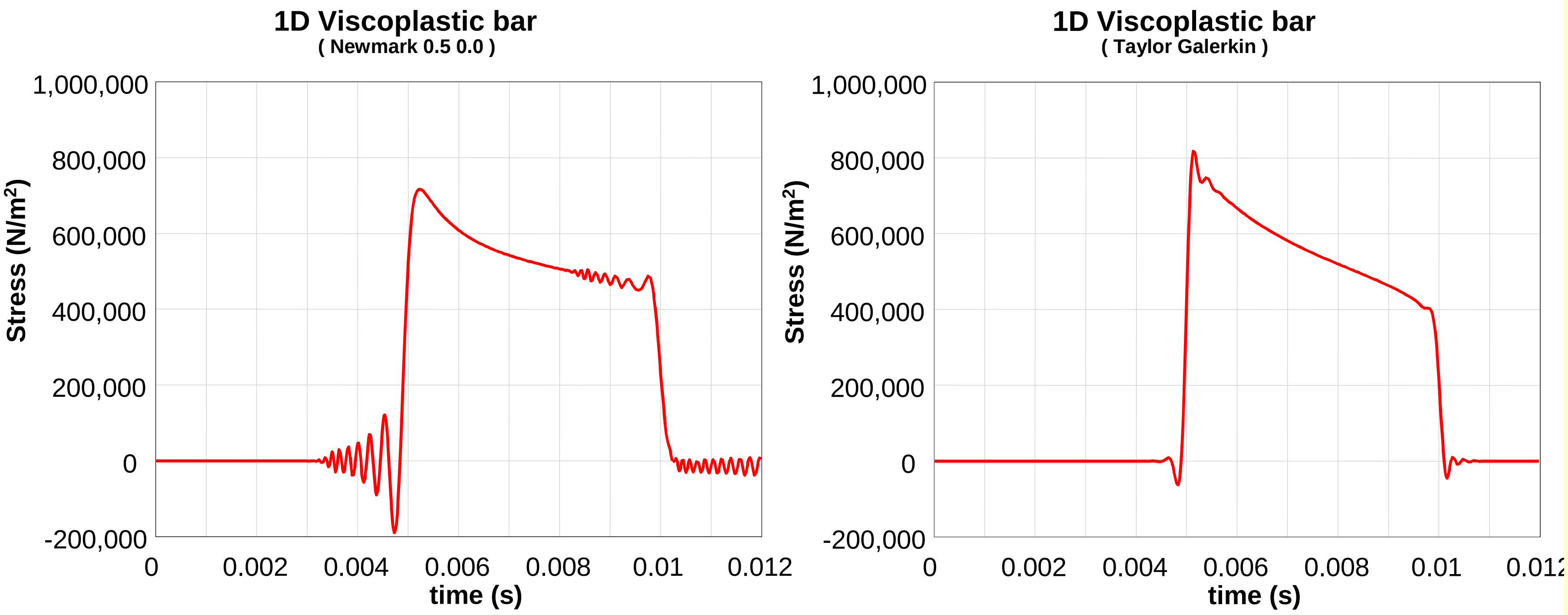


Velocity at at x = L/4 (Newmark vs Taylor Galerkin)



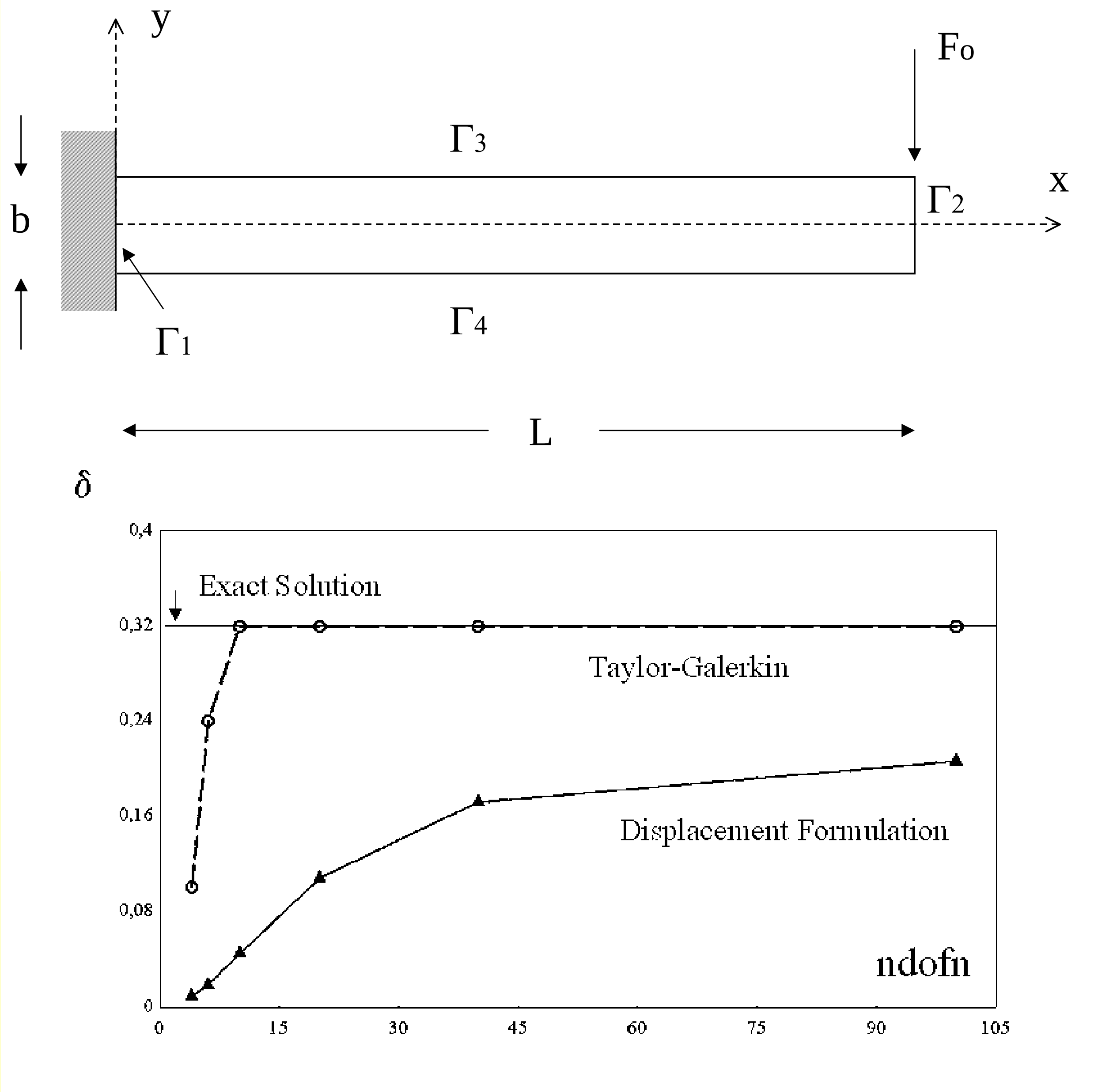


Stress at left end (Newmark vs Taylor Galerkin)

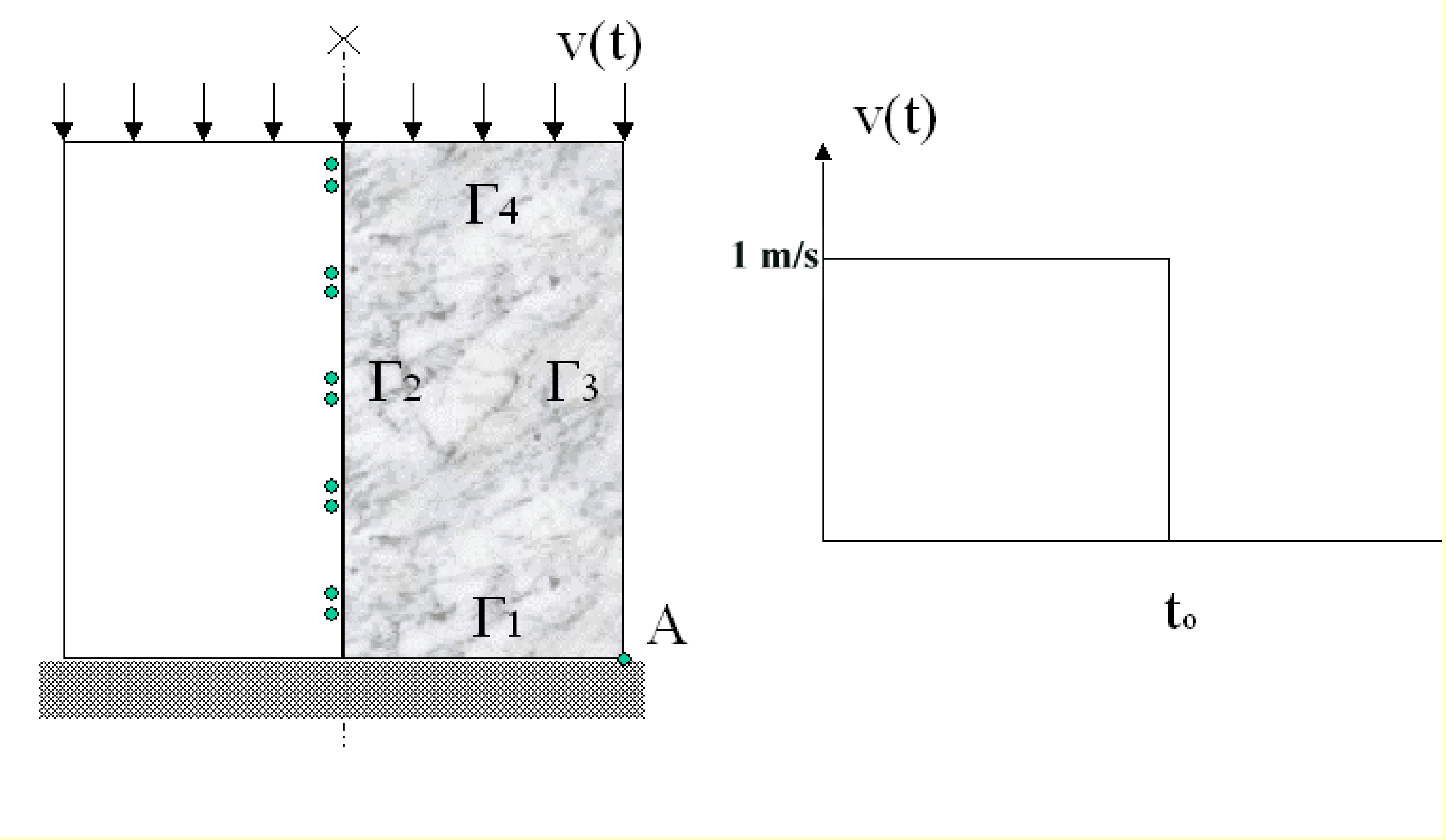


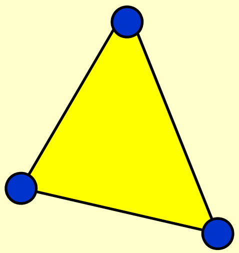
Stress at left end (Newmark vs Taylor Galerkin)

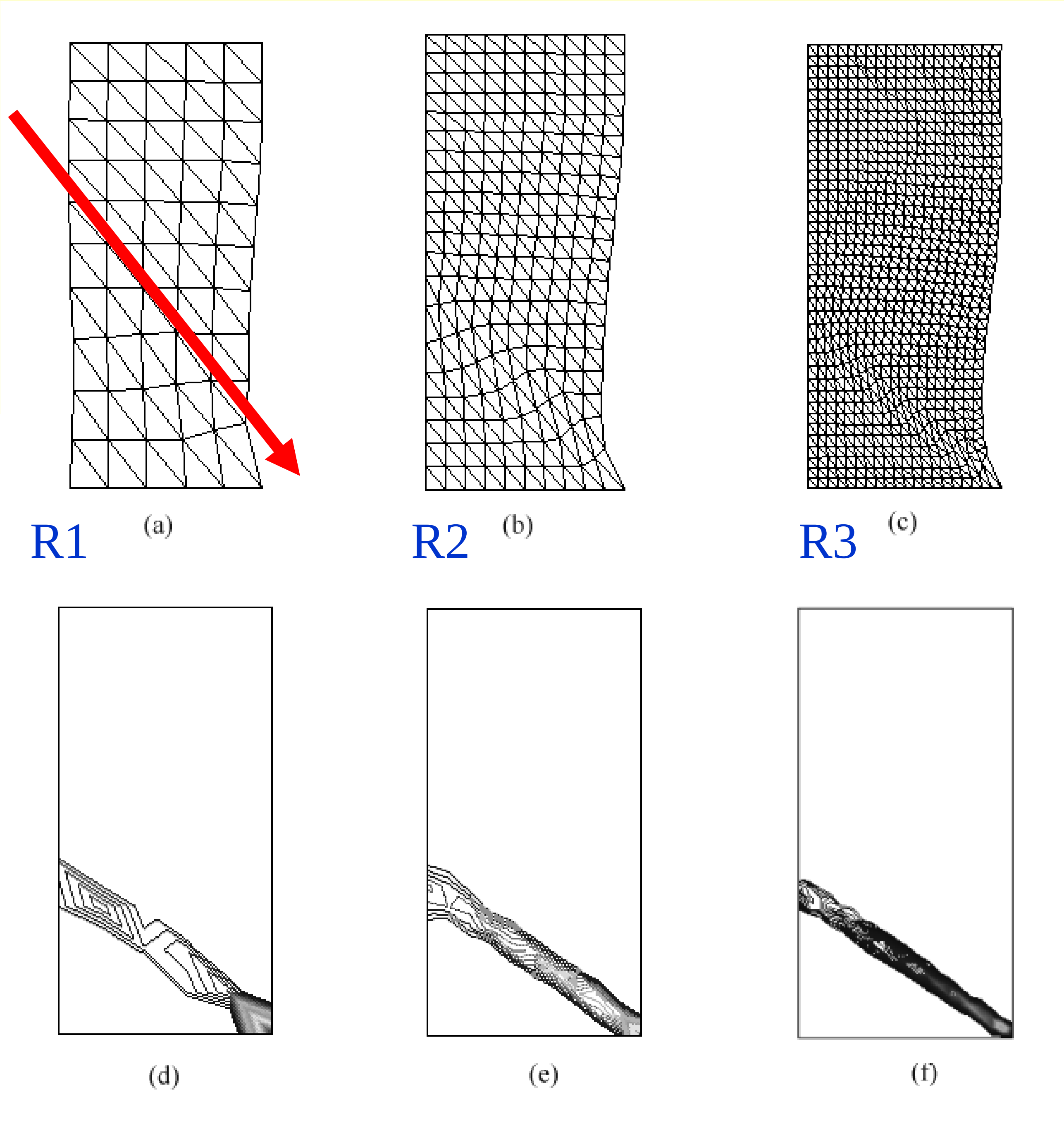
Example: Bending of a Cantilever Beam

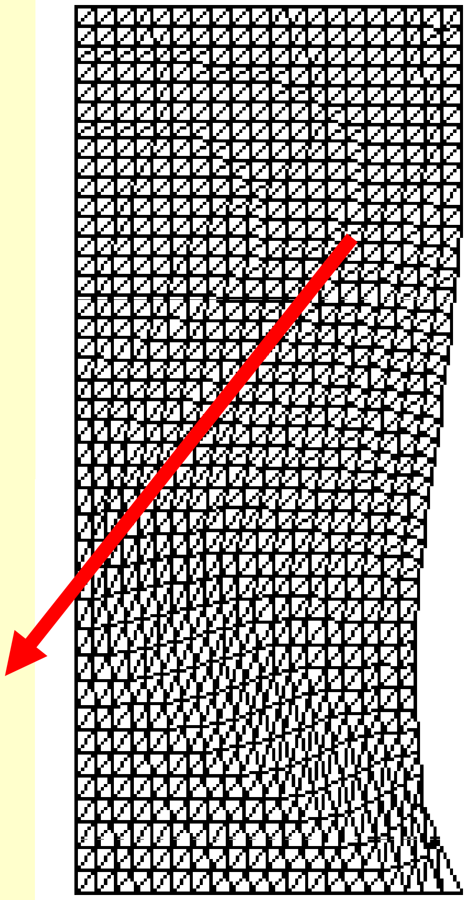


Example: 2D Localization

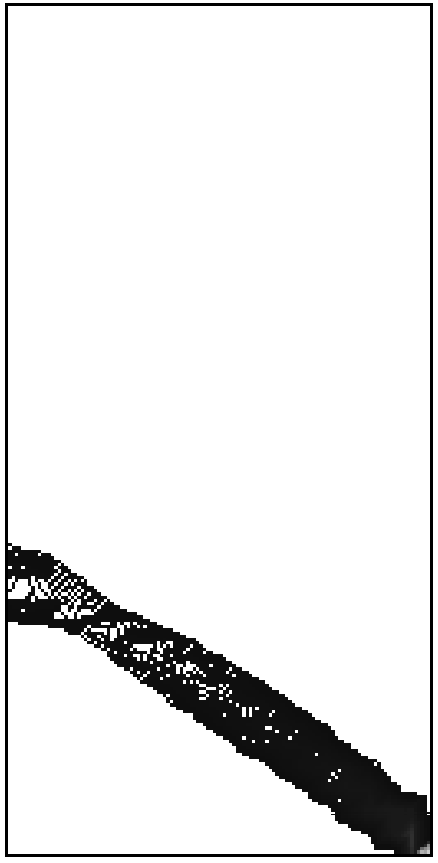


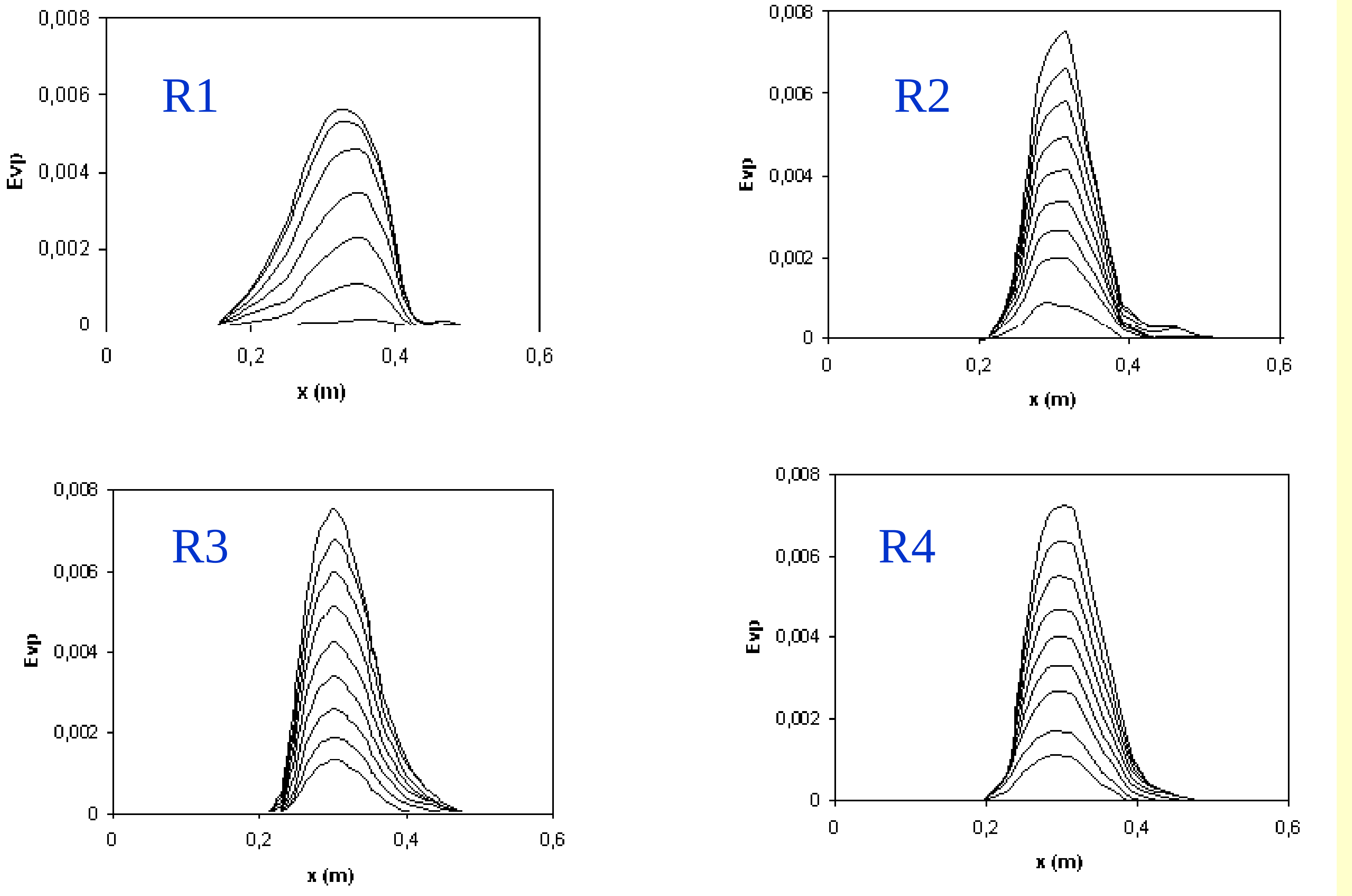



R1
(a)
R2
(b)
R3
(c)
(d)
(e)
(f)



w1




R1
R2
R3
R4
0,008
0,006
0,004
0,002
0
Evp
0
0,2
0,4
0,6
x (m)

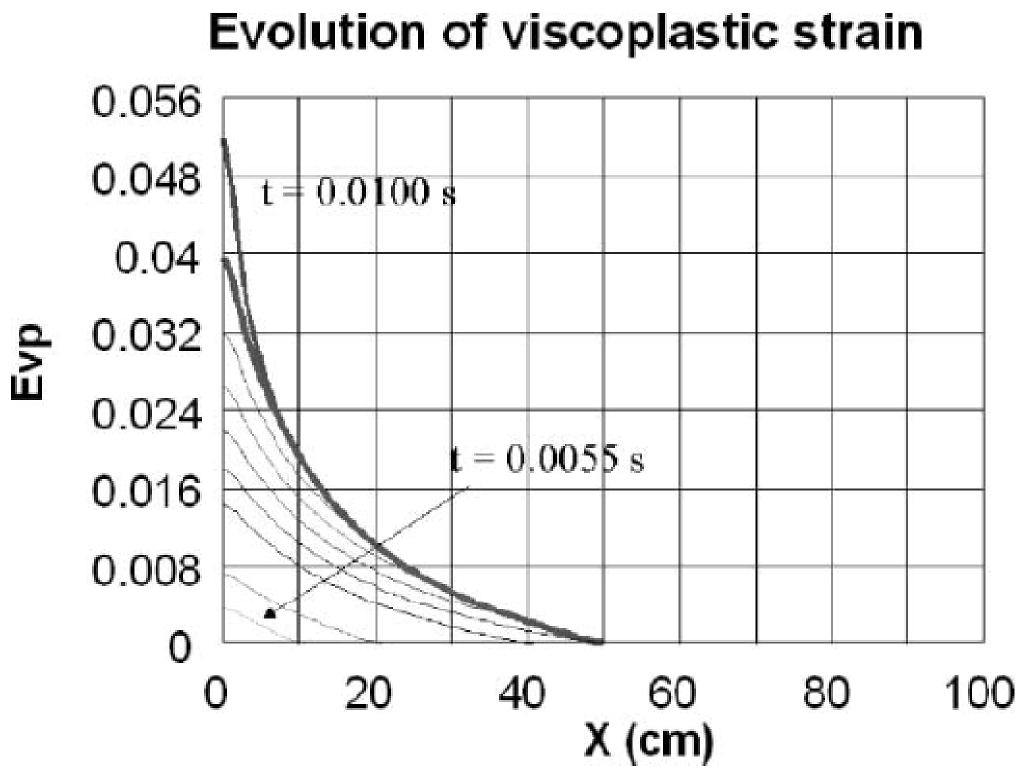


Fig. 6. Viscoplastic strain along the bar for different times.

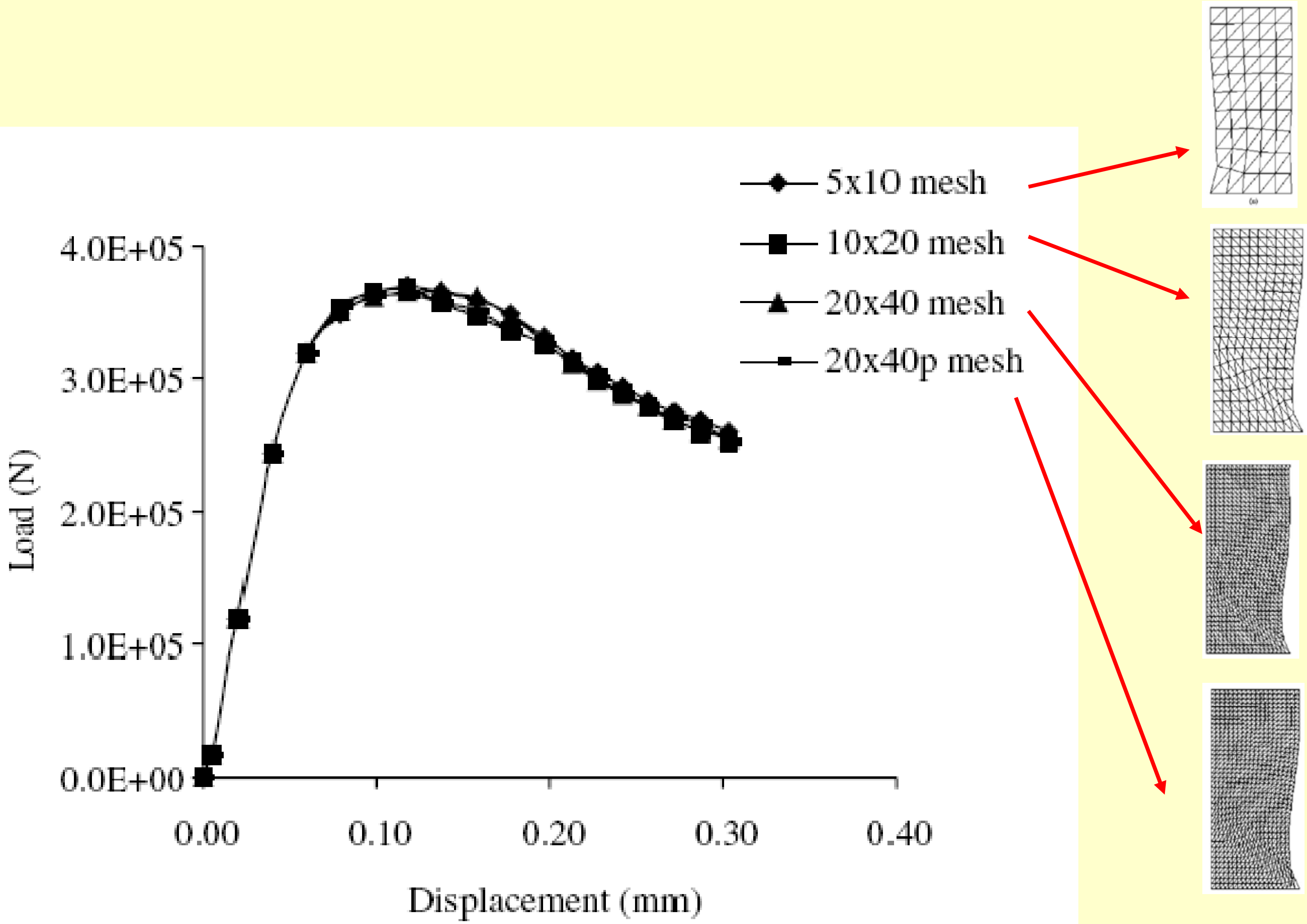
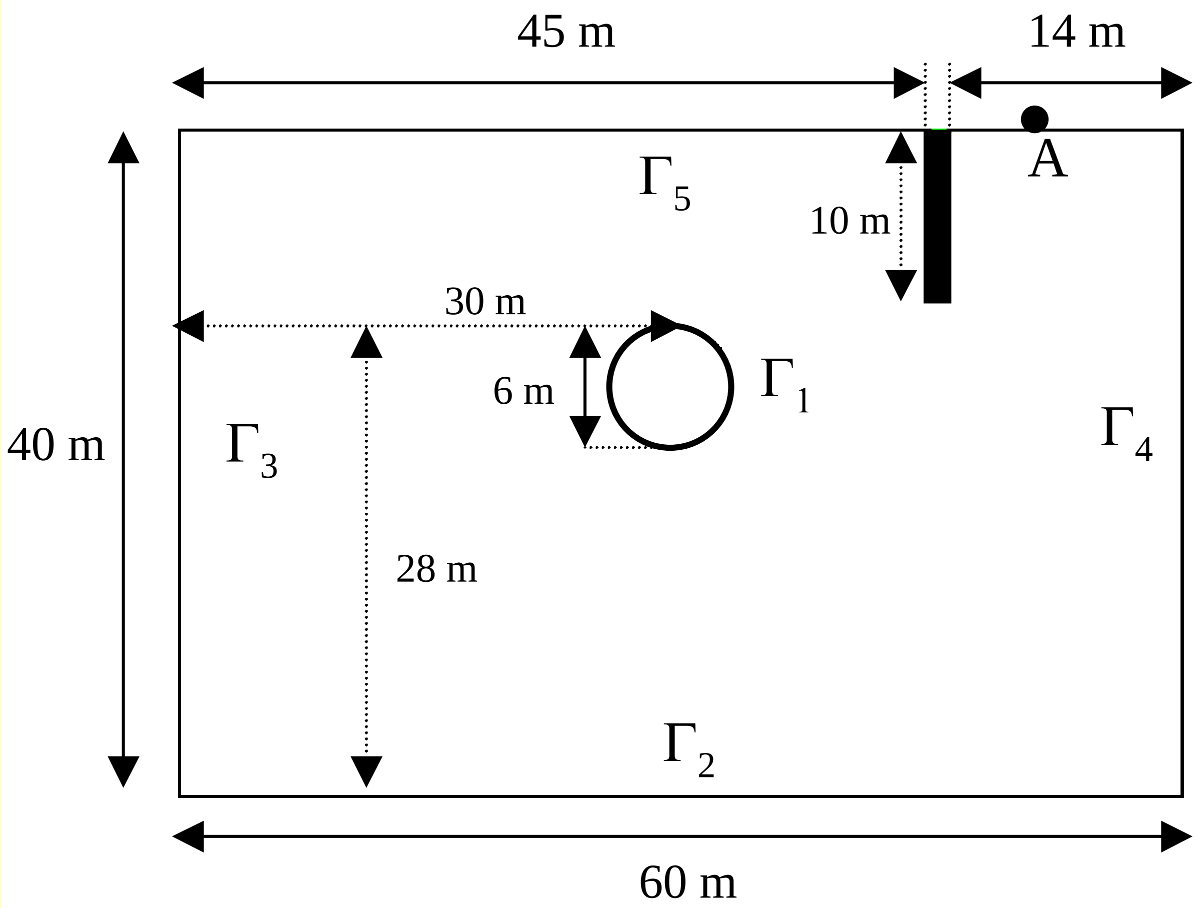
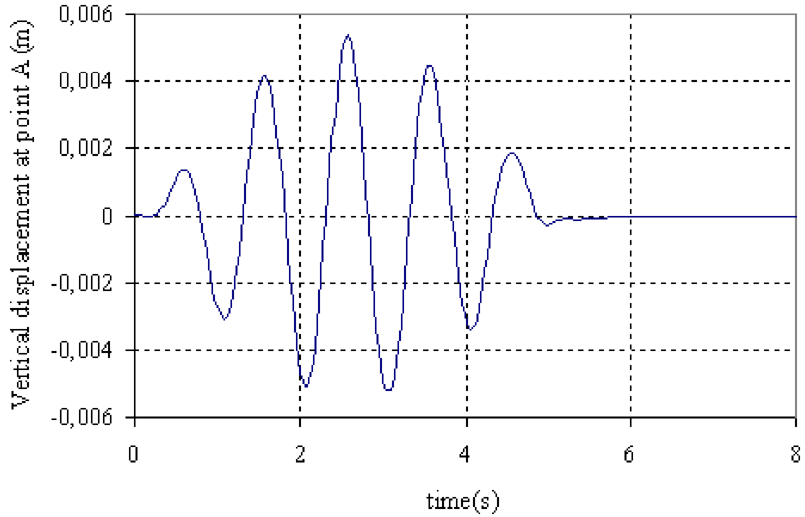


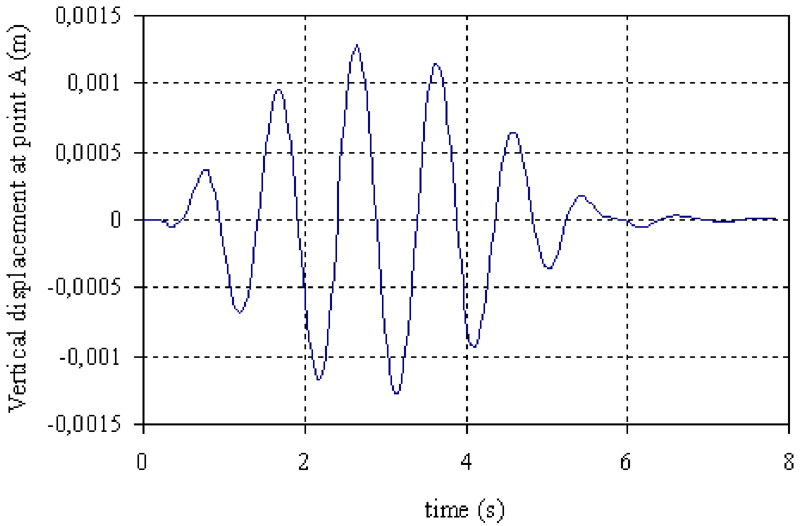
Fig. 13. Load–displacement curves.

Example: Vibration transmission




0,006
0,004
0,002
0
-0,002
-0,004
-0,006
0
2
4
6
8
Vertical displacement at point A (m)
time(s)

(a)


0,0015
0,001
0,0005
0
-0,0005
-0,001
-0,0015
0
2
4
6
8
Vertical displacement at point A (m)
time (s)

(b)

1st order equations: coupled problems

$$\rho \frac{\partial v}{\partial t} = \operatorname{div} \sigma + \rho b$$

$$\frac{\partial \sigma}{\partial t} = D^e \operatorname{grad}^s v - D^e \dot{\varepsilon}^{vp}$$

$$\rho \frac{\partial v}{\partial t} = \operatorname{div} \sigma' - \underline{\operatorname{grad} p_w} + \rho b$$

$$\frac{\partial \sigma}{\partial t} = D^e \operatorname{grad}^s v - D^e \dot{\varepsilon}^{vp}$$

$$\frac{1}{Q^*} \frac{dp_w}{dt} = \operatorname{div}\left(k \operatorname{grad} p_w\right) - \operatorname{div} v$$

Incompressible, impermeable limit

$$\cancel{\frac{1}{Q^*} \frac{dp_w}{dt}} = \operatorname{div}\left(\cancel{k \operatorname{grad} p_w}\right) - \operatorname{div} v$$

$$\operatorname{div} v = 0$$

Stabilization with fractional step

$$\rho \frac{\partial v}{\partial t} = -\mathrm{grad}\, p + b$$
$$\mathrm{div}\, v = 0$$

$$\Rightarrow$$

$$\rho \frac{v^* - v^n}{\Delta t} = b$$
$$\rho \frac{v^{n+1} - v^*}{\Delta t} = -\mathrm{grad}\, p^{n+1}$$
$$\mathrm{div}\, v^{n+1} = 0$$

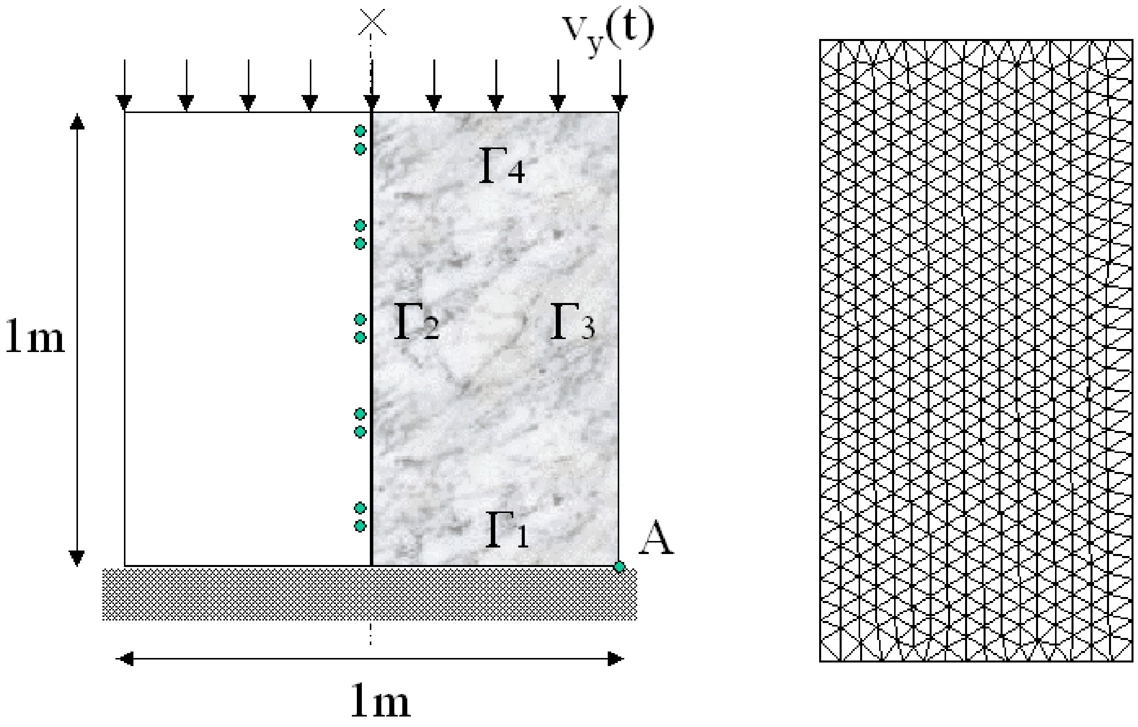
$$\mathrm{div}\left(\frac{v^{n+1} - v^*}{\Delta t}\right) = -\frac{1}{\rho}\mathrm{div}\left(\mathrm{grad}\, p^{n+1}\right) \Rightarrow \mathrm{div}\, v^* = -\frac{\Delta t}{\rho}\nabla^2 p^{n+1}$$

S1 $$\rho \frac{v^* - v^n}{\Delta t} = b$$

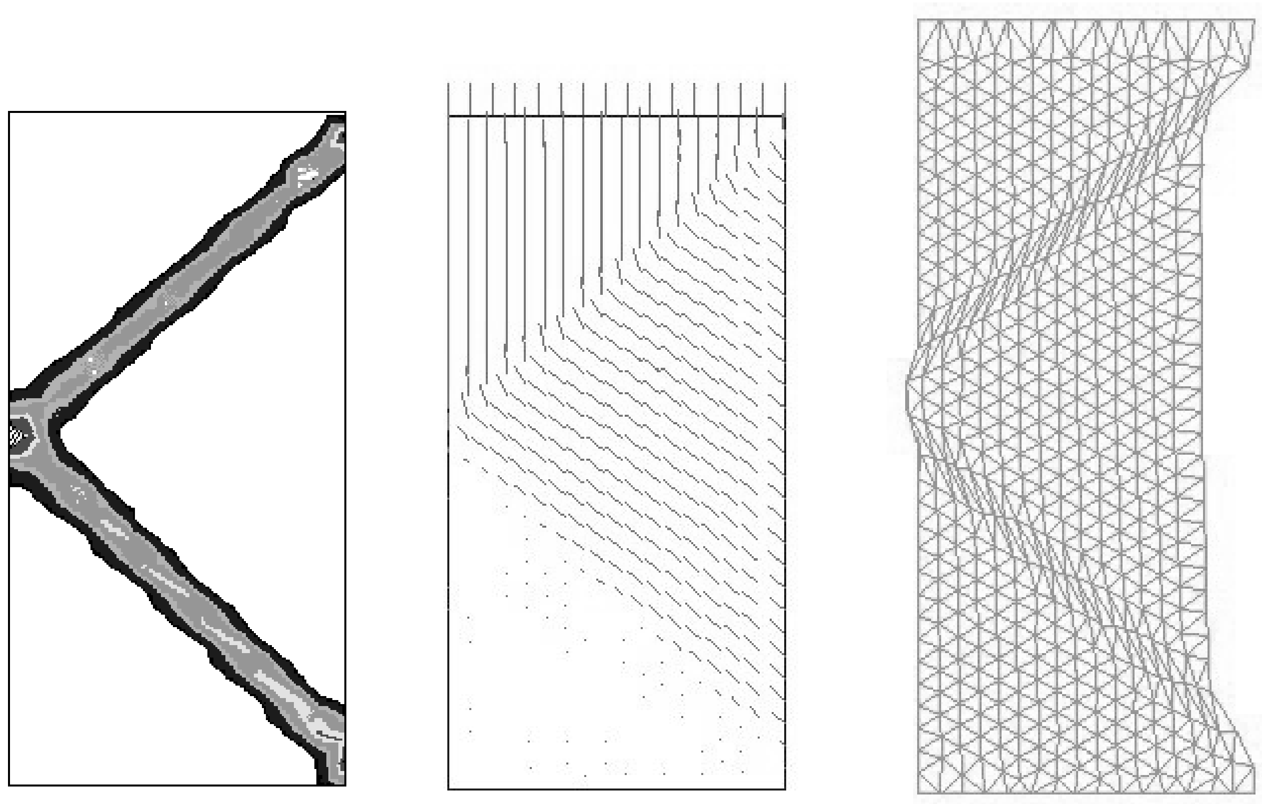
S2 $$\mathrm{div}\, v^* = -\frac{\Delta t}{\rho}\nabla^2 p^{n+1}$$

S3 $$\rho \frac{v^{n+1} - v^*}{\Delta t} = -\mathrm{grad}\, p^{n+1}$$

Application: localization in a Cam Clay viscoplastic specimen



(a) (b)



(a) (b) (c)

Contents

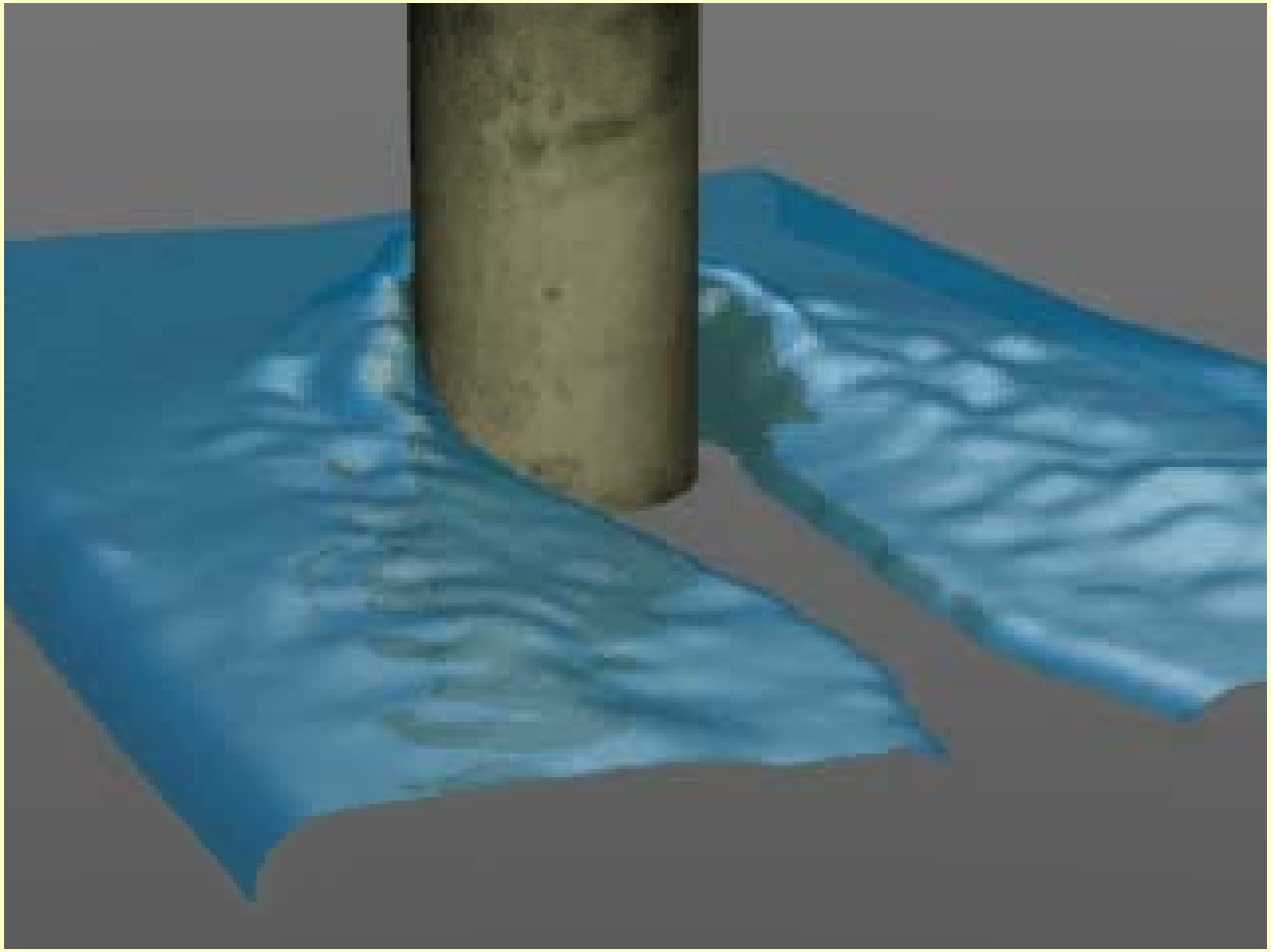
SPH: what is it?

- Used in
  - CFD
  - Animation
  - Solid dynamics
  - Soil mechanics
  - …..

- Advantages
  - Lagrangian
  - Large deformation
  - Multiphysics
  - …..

- Disadvantages
  - Boundaries
  - Searching neighbours
  - Explicit
  - …..

- Smooth Particle Hydrodynamics



Benchmarks and applications

- Vertical cut under gravity (Von Mises)

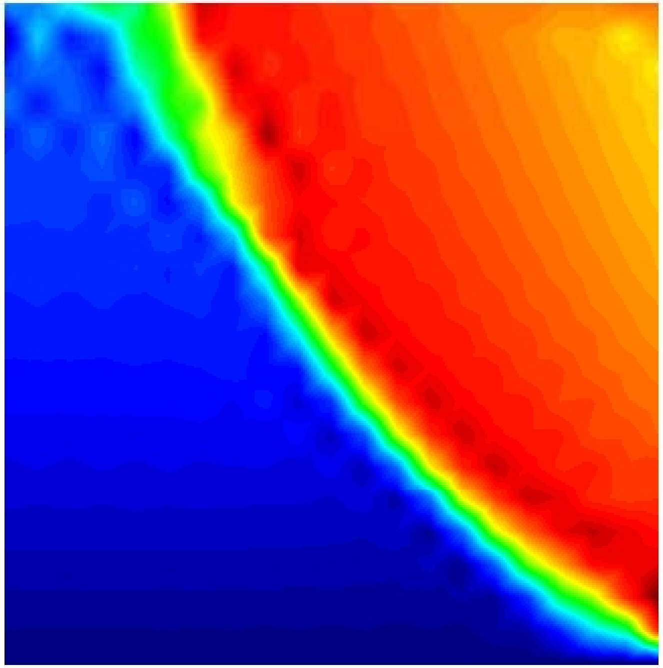
$E = 8 \cdot 10^7$ Pa

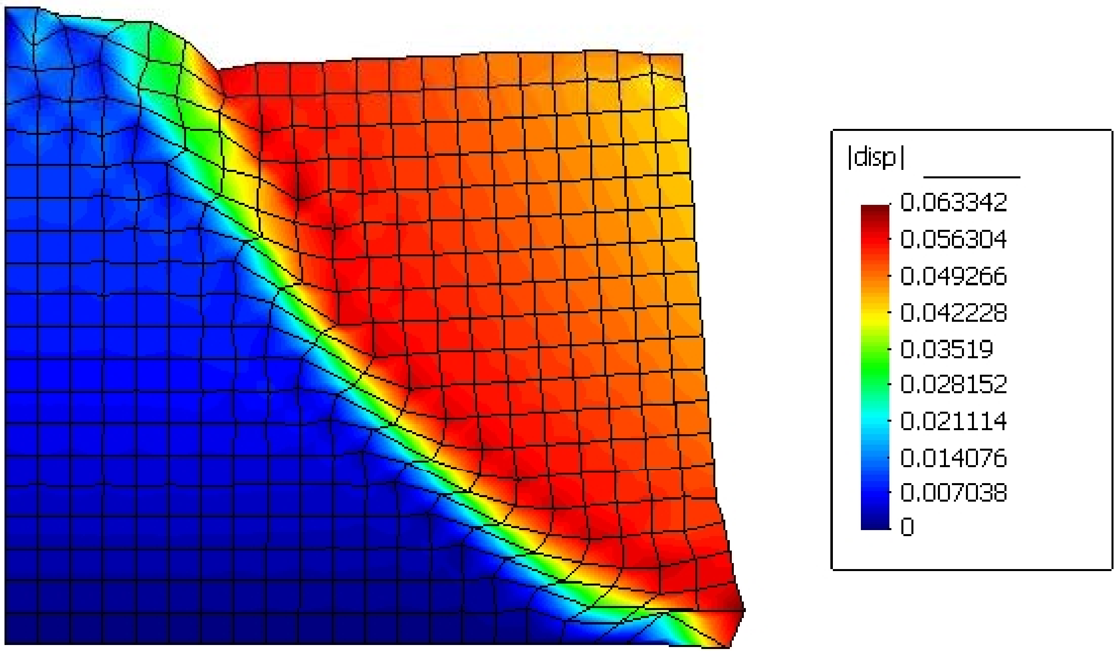
$\nu = 0.3$

$\rho = 2000\ kg / m^3$

$\gamma = 2$ s^{-1} $\quad N = 1$

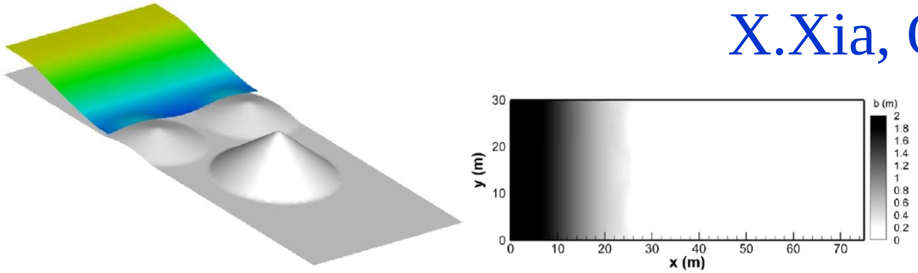
$\sigma_0 = 125000$ Pa $\quad H = -8 \cdot 10^6$ Pa



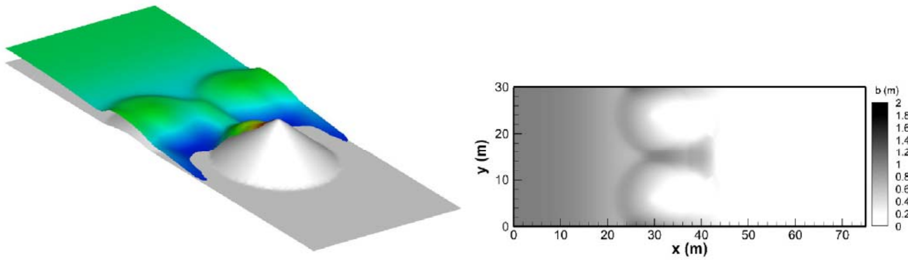


X.Xia, Q.Liang, M.Pastor,W.Zou, YFZhang

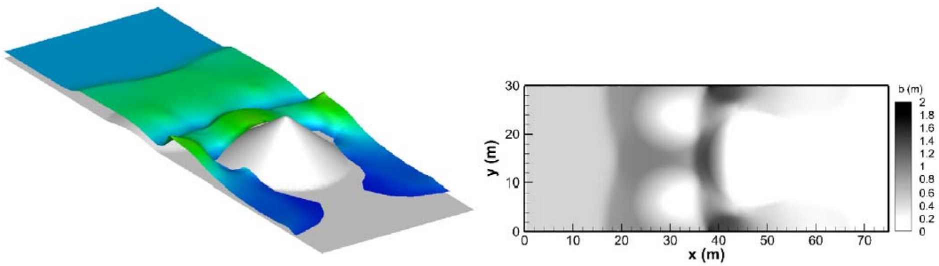
Advances in Water Resources
59 (2013) 25–38



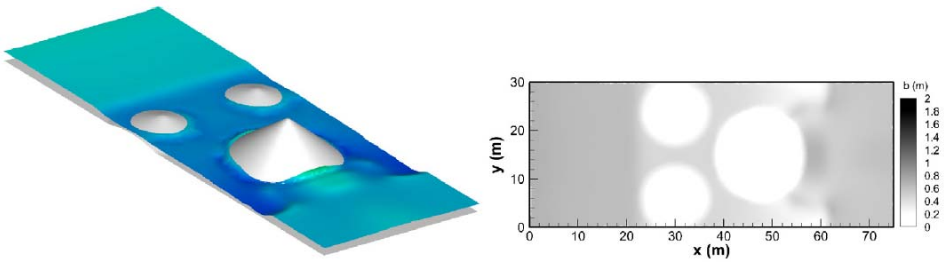
(a)



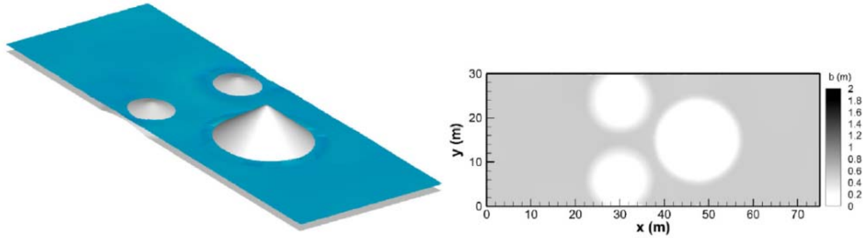
(b)



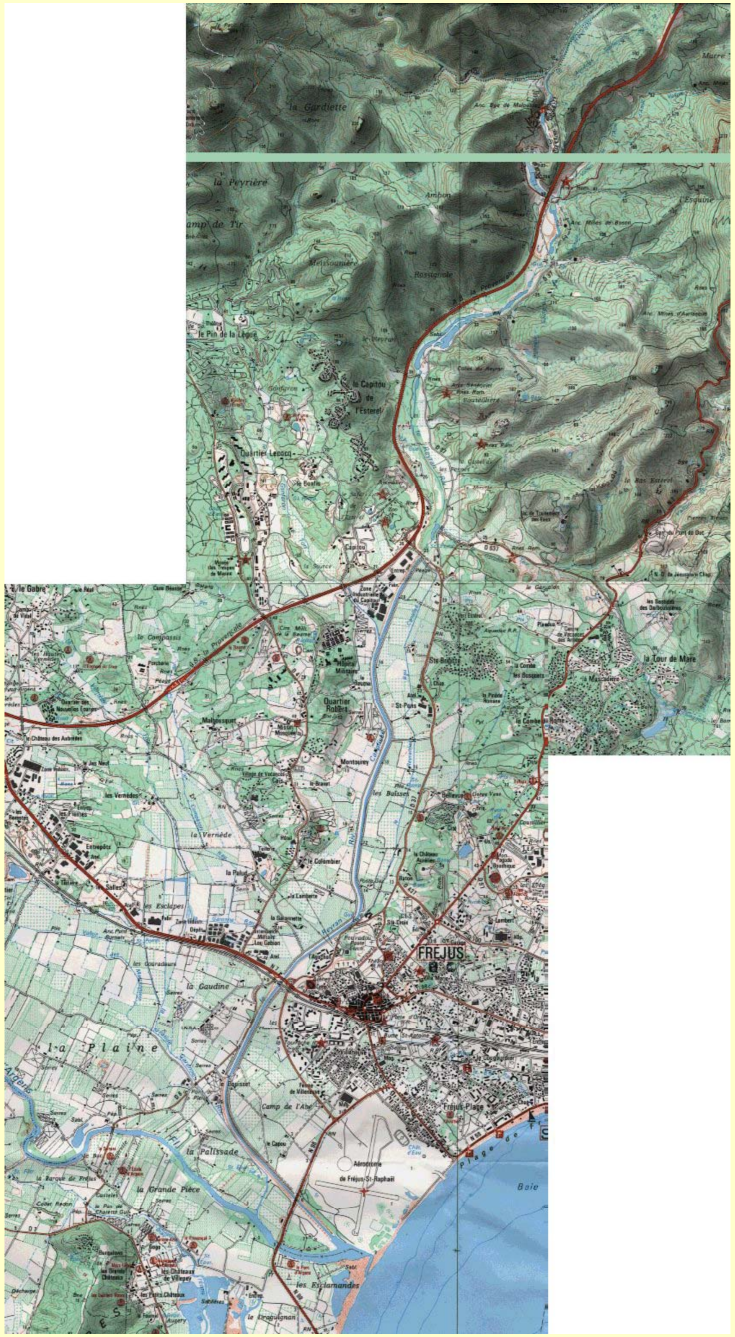
(c)



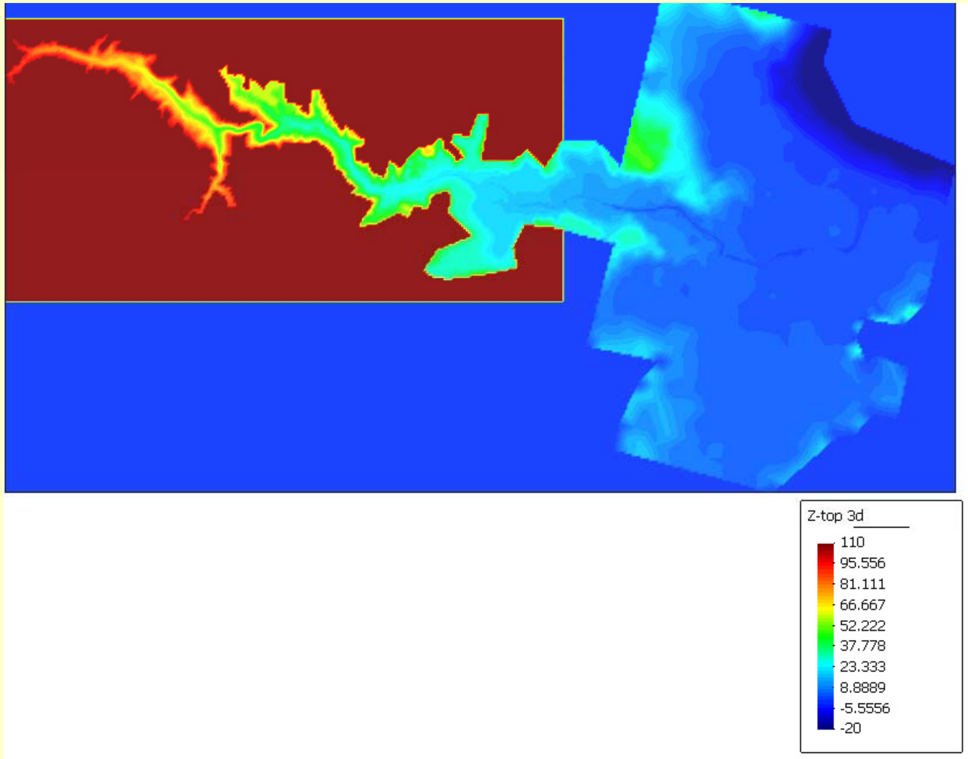
(d)



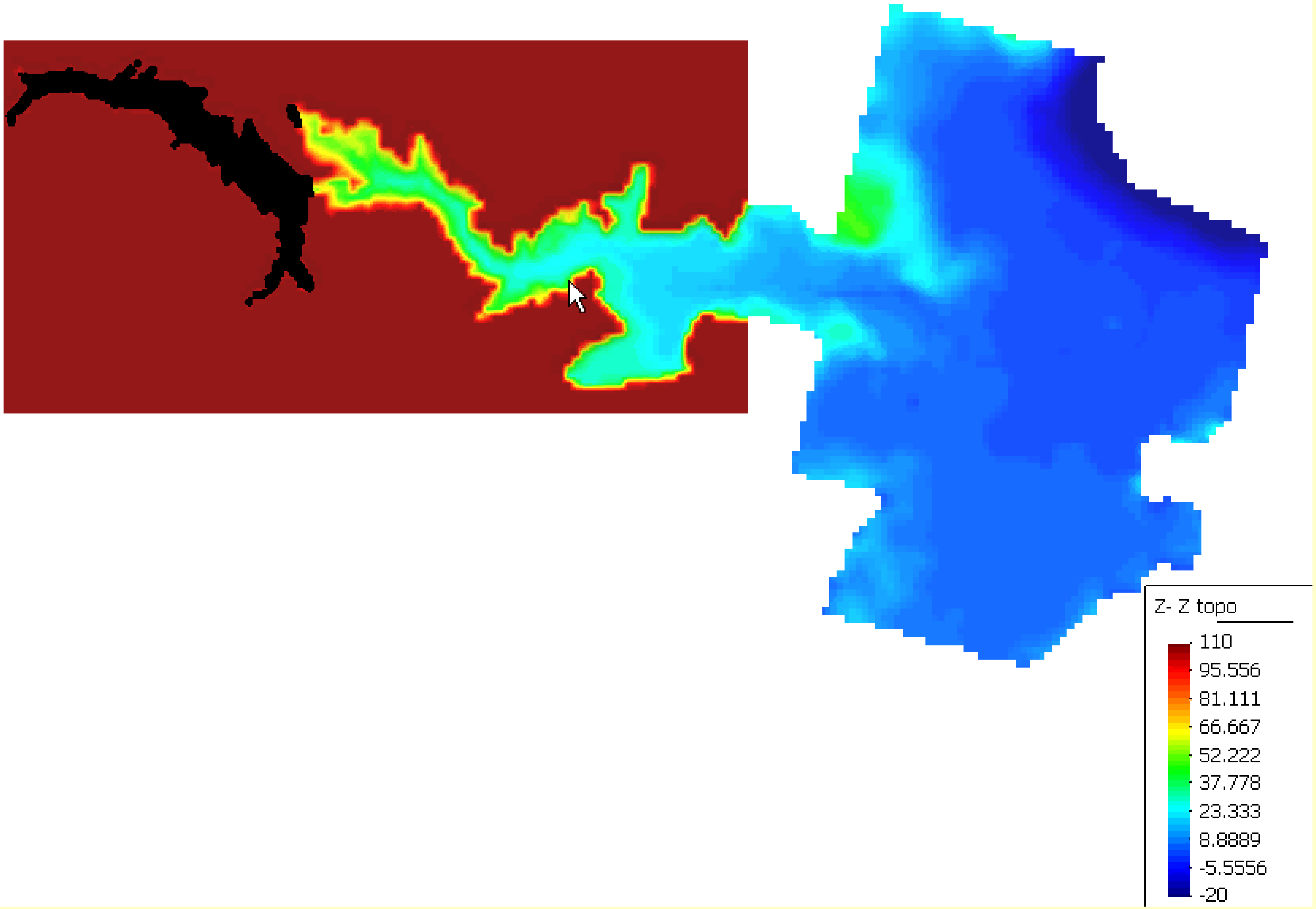
(e)



Malpasset dam failure: December 1959



Madrid + Newcastle University
Prof. Qiuhua Liang, Yueling Huang



Z- Z topo
110
95.556
81.111
66.667
52.222
37.778
23.333
8.8889
-5.5556
-20

Based on:

- T.Blanc, M.Pastor
  A stabilized Smoothed Particle Hydrodynamics, Taylor-Galerkin algorithm
  for soil dynamics problems
  Int. J. Numer. Anal. Meth. Geomech. (2011)
  Volume 37, Issue 1, pages 1–30, January 2013

- T.Blanc, M.Pastor
  A stabilized Fractional Step, Runge Kutta Taylor SPH algorithm for coupled problems in
  Geomechanics
  Comput. Methods Appl. Mech. Engrg. Volumes 221–222, 1 May 2012, Pages 41–53

- T.Blanc, M.Pastor
  A stabilized Runge Kutta, Taylor Smoothed Particle Hydrodynamics algorithm
  for large deformation problems in dynamics.
  Int.J.Num.Meth.Engineering
  Volume 91, Issue 13, pages 1427–1458, 28 September 2012

Based on:

- M. Pastor, M. Martin Stickle, P. Dutto, P. Mira, J.A. Fernández Merodo, T. Blanc, S.Sancho, A.S. Benítez
  A viscoplastic approach to the behaviour of fluidized geomaterials with application to fast landslides
  Continuum Mech. Thermodyn. (2013)

- M.Pastor, B.Haddad, G.Sorbino, S.Cuomo and V.Drempetic
  A depth- integrated, coupled SPH model for flow-like landslides and related phenomena
  Int. J. Numer. Anal. Meth. Geomech 2009; 33: 143-172 doi: 10.1002/nag.075

Contents

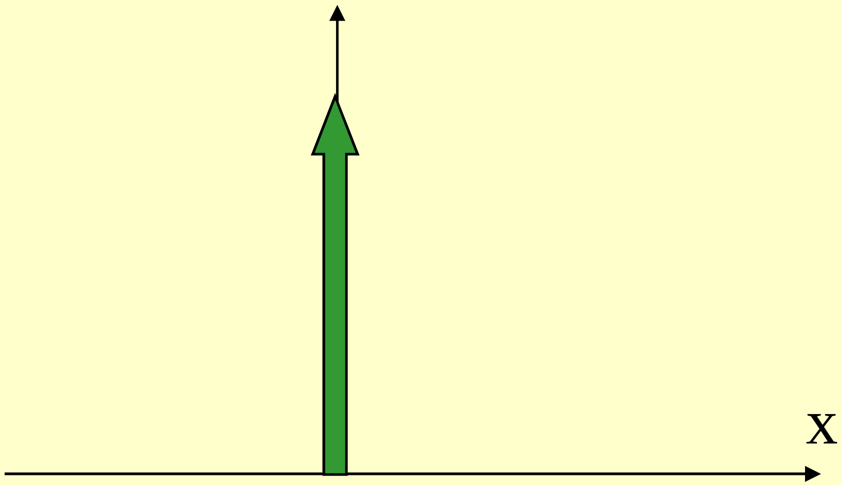
- Introduction
- SPH (I) Basic Model
- SPH (II) Large deformation
- SPH (III) Coupled v-p_w
- SPH (IV) Depth integrated models h-v-p_w

SPH (I) Basic Model

- Basic identity

$$\boxed{\phi(x) = \int_{\Omega} \phi(x')\delta(x'-x)\,dx'}$$

- Dirac's Delta



$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & |x| > 0 \end{cases}$$

$$\int_{\Omega} \delta(x)\,dx = 1$$

- Dirac's Delta is a singular distribution

$$\delta[\phi] = \phi(0)$$

- Distributions or generalized functions:
  linear continuous functionals

$$T_W[\phi] = \int_\Omega \boxed{W(x')}\,\boxed{\phi(x')}\,dx'$$

Test function

Kernel

- Singular distributions

$$\delta[\phi] = \phi(0)$$

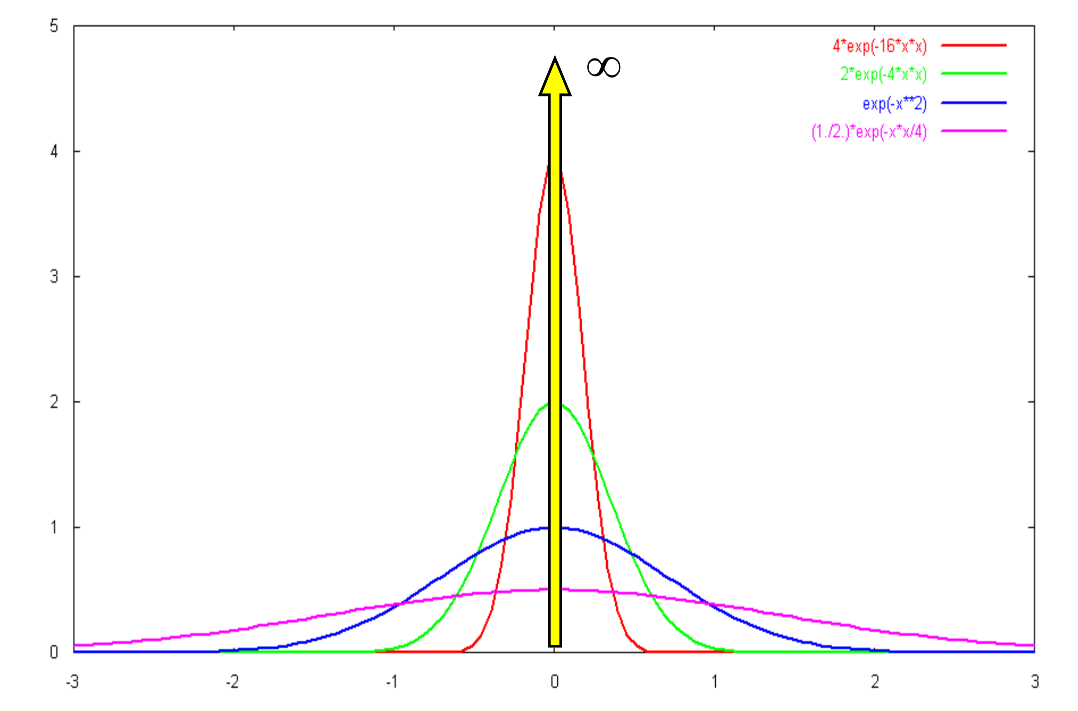
Can be obtained as limits of series of regular distributions

- We build the series

$$T_{W\,k}[\phi] = \int_{\Omega} W_k(x',h)\,\phi(x')\,dx'$$

$$W_k(x,h) = \frac{1}{h\sqrt{\pi}} \exp\left(-\frac{x^2}{h^2}\right)$$

with $h = 1/k$



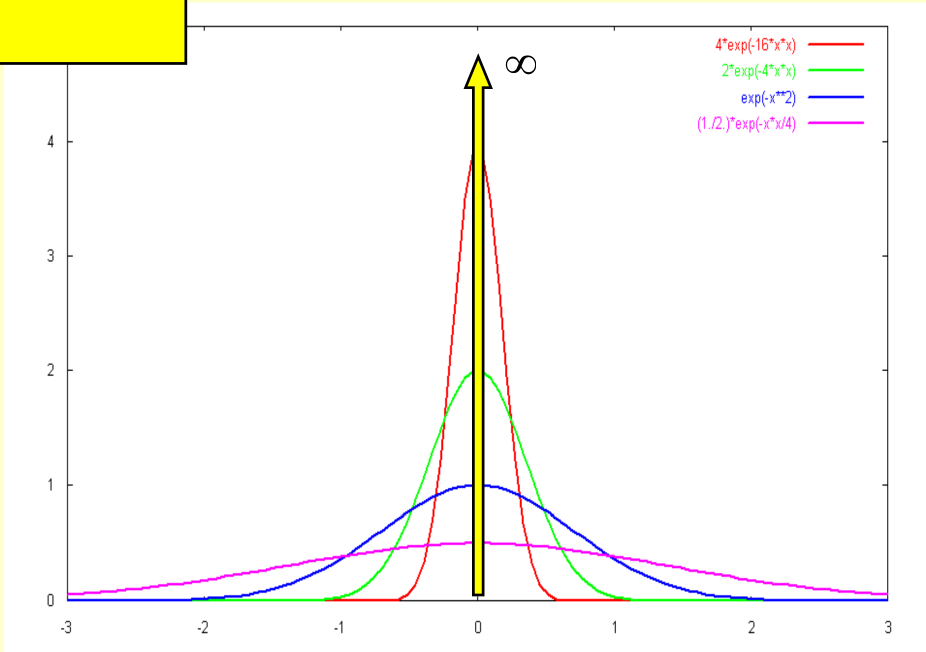
$$\lim_{k\to\infty} T_{W\,k}[\phi] = \phi(0)$$

$$\delta[\phi] = \phi(0)$$

$$\boxed{\lim_{k\to\infty} T_{W\,k}[\phi] = \delta[\phi]}$$

- Integral approximation of functions

$$< \phi(x) > \approx \int_{\Omega} \phi(x') W(x' - x, h)\, dx'$$



- Kernel properties

$$\lim_{h \to 0} W(x' - x, h) = \delta(x)$$

$$\int_{\Omega} W(x' - x, h)\, dx' = 1$$

$$W(x' - x, h) = 0 \;\; if \quad |x' - x| \geq kh$$

Monotonous decreasing function of *x*

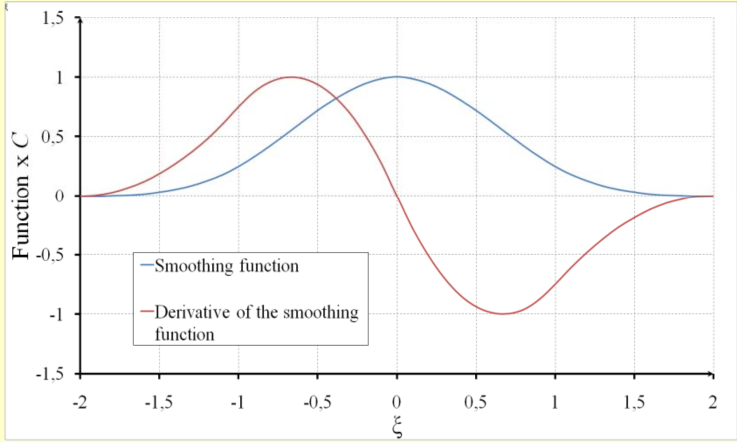
Numerical model – Smoothed Particle Hydrodynamics (SPH)

- Numerical meshless method based on integral approximations (kernel approximation) and approximations on set of discrete points *(Gingold and Monaghan 1977 ; Lucy 1977)*

- Integral approximation of a function ϕ using a kernel W

$$\phi(x) \approx \int_{\Omega} \phi(x') W(x - x', h)\, dx'$$

h is the smoothing length



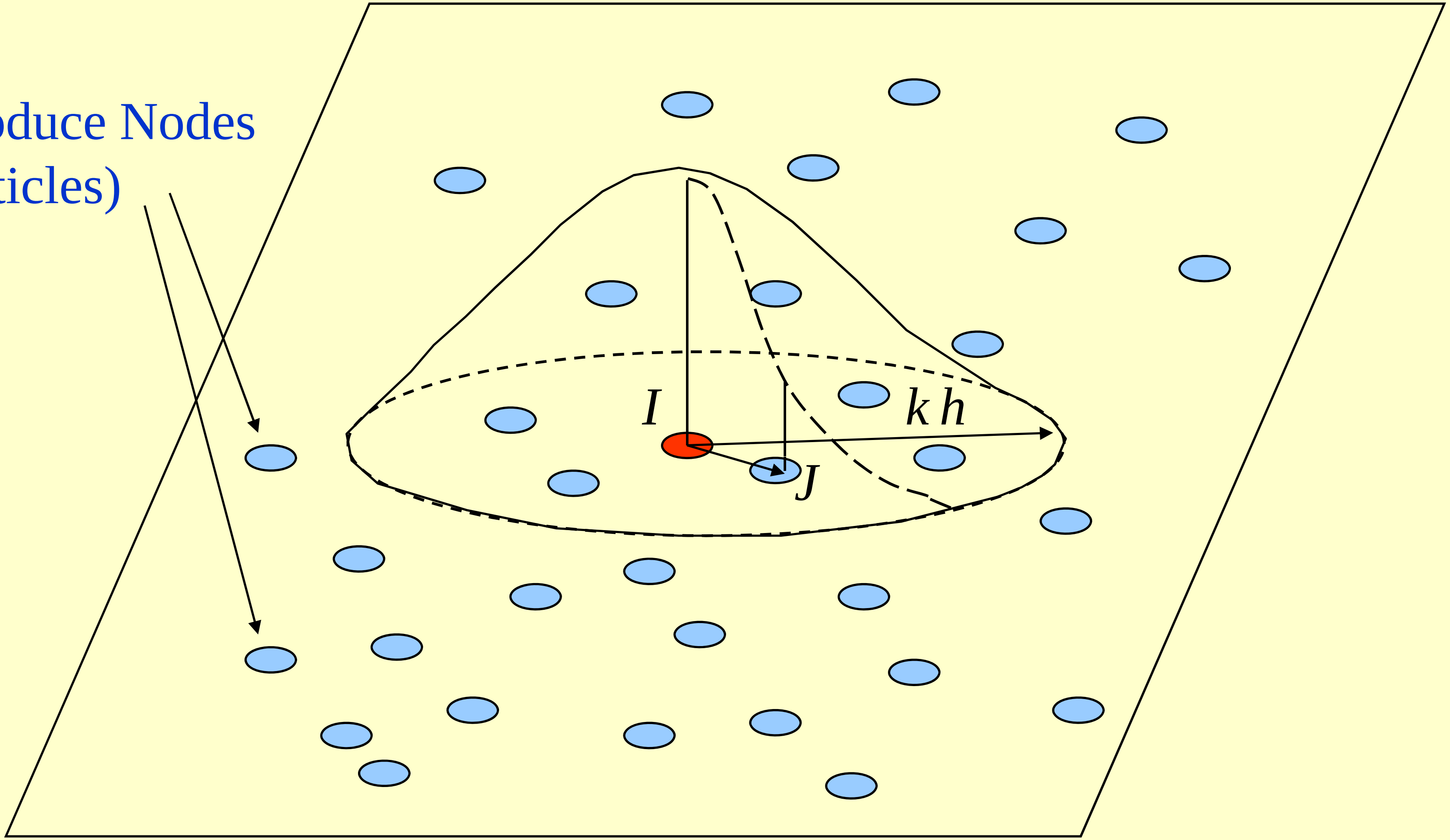
- Integral approximation of the derivative of a function ϕ

$$\operatorname{grad}\phi(x) \approx -\int_{\Omega} \phi(x') \cdot \operatorname{grad} W(x - x', h)\, dx'$$

- SPH discretization of Integral Approximations (Functions)

$$< \phi(x) > = \int_{\Omega} \phi(x')\, W(x' - x, h)\, dx'$$

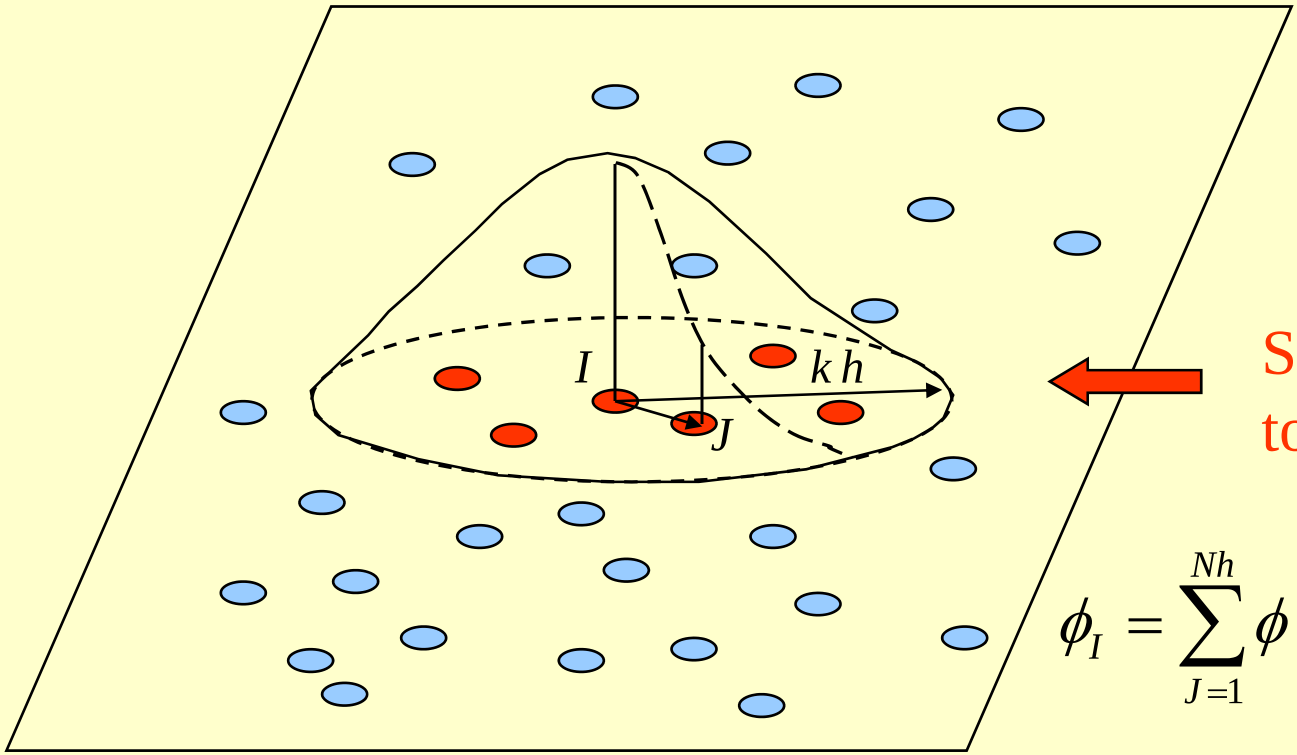
Introduce Nodes (Particles)



- Numerical Integration

$$<\phi(x)> = \int_\Omega \phi(x')W(x'-x,h)\,dx'$$

$$\phi_I = \langle \phi(x_I) \rangle_h = \sum_{J=1}^{N} \phi(x_J)W(x_J - x_I, h)\,\omega_J$$



Summation extended to nodes within *kh*

$$\phi_I = \sum_{J=1}^{Nh} \phi(x_J)W(x_J - x_I, h)\,\omega_J$$

- Properties of SPH approximations

$$\langle \alpha \phi \rangle_h = \alpha \langle \phi \rangle_h \qquad \alpha \in \mathbb{R}$$

$$\langle \phi + \psi \rangle_h = \langle \phi \rangle_h + \langle \psi \rangle_h$$

$$\langle \phi \psi \rangle_h = \langle \phi \rangle_h . \langle \psi \rangle_h$$

$$\left\langle \frac{d\phi(x,t)}{dt} \right\rangle_h = \frac{d}{dt} \langle \phi \rangle_h$$

Example

$$\frac{D\rho}{Dt} + \rho \operatorname{div} v = 0$$

$$\frac{D}{Dt} \langle \rho \rangle + \langle \rho \rangle \langle \operatorname{div} v \rangle = 0$$

$$\frac{D\rho_I}{Dt} = -\rho_I \sum_J \frac{m_J}{\rho_J} v_J \operatorname{grad} W_{IJ}$$

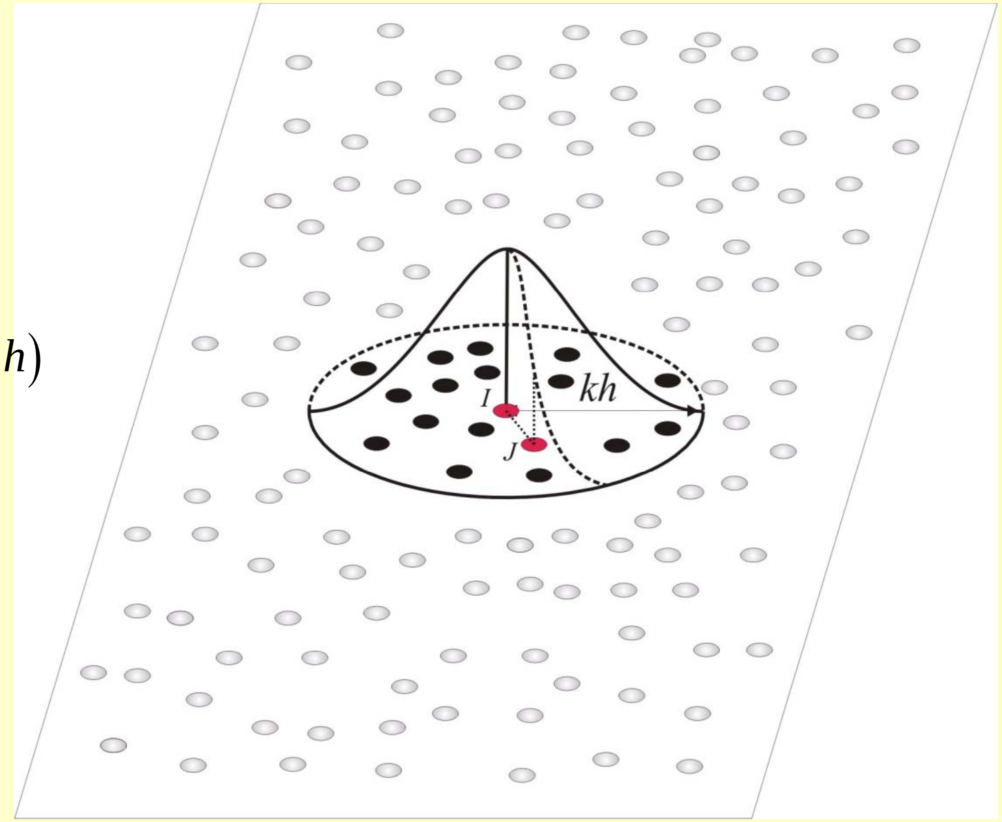
Numerical model – Smoothed Particle Hydrodynamics (SPH)

- Particle approximation

$$\phi_I = \sum_{J=1}^{N_P} \frac{m_J}{\rho_J} \phi(x_J) W_{IJ}$$

with $W_{IJ} = W(x_I - x_J, h)$

$$\mathrm{grad}\,\phi_I = \sum_{J=1}^{N_P} \frac{m_J}{\rho_J} \phi(x_J)\,\mathrm{grad}\,W_{IJ}$$



Numerical model – Smoothed Particle Hydrodynamics (SPH)

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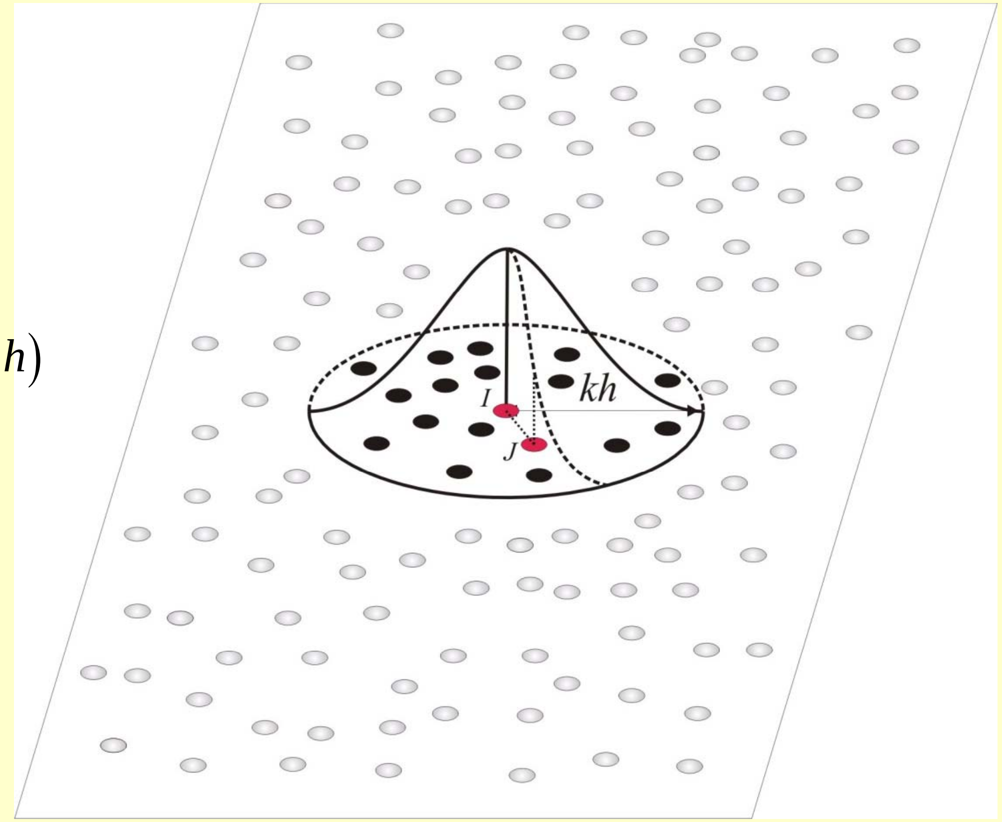


- Boundary deficiency problem ➔ Normalization of the approximations (Chen 1999)

Boundary deficiency

TSPH d1 Corrected SPH

1D

$$\phi_I = \frac{\sum_{J=1}^{N_P} \frac{m_J}{\rho_J} \phi(\boldsymbol{x_J}) W_{IJ}}{\sum_{J=1}^{N_P} \frac{m_J}{\rho_J} W_{IJ}}$$

I,J refer to SPH nodes

$$W_{IJ} = W(\boldsymbol{x_I} - \boldsymbol{x_J}, h)$$

2D

$$A\boldsymbol{\phi} = \boldsymbol{r} \qquad A = \begin{pmatrix} A_{11,I} & A_{12,I} \\ A_{21,I} & A_{22,I} \end{pmatrix} \quad \boldsymbol{\phi} = \begin{pmatrix} \phi_{1,I} \\ \phi_{2,I} \end{pmatrix} \quad \boldsymbol{r} = \begin{pmatrix} r_{1,I} \\ r_{2,I} \end{pmatrix}$$

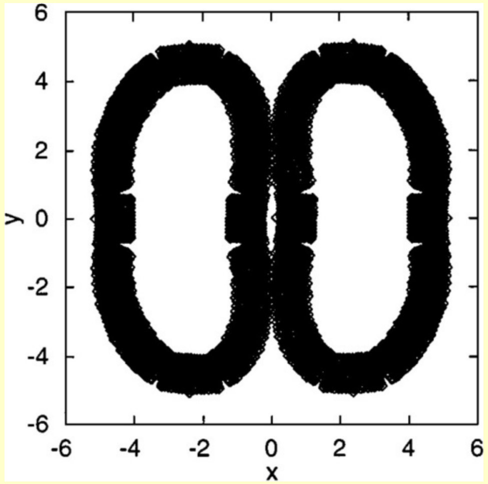
$$A_{\beta\alpha,I} = \sum_{J=1}^{N_P} \frac{m_J}{\rho_J} \left(\boldsymbol{x_{\alpha,J}} - \boldsymbol{x_{\alpha,I}} \right) W_{IJ,\beta}$$

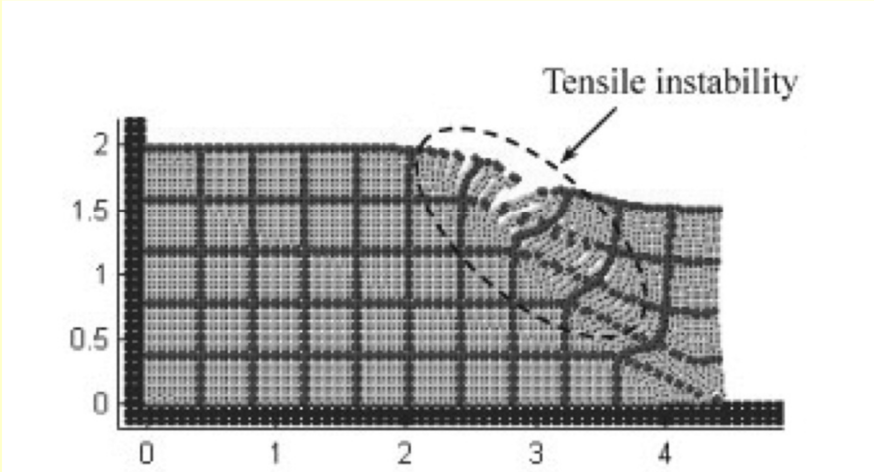
$$r_{\beta,I} = \sum_{J=1}^{N_P} \frac{m_J}{\rho_J} \left[\phi(\boldsymbol{x_J}) - \phi(\boldsymbol{x_I}) \right] W_{IJ,\beta}$$

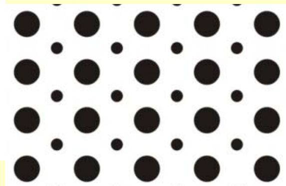
Numerical model – SPH tensile instability

- SPH presents a tensile instability for dynamics problems with material strength (Li and Liu 2003)

- Results in particle clumping y collapse of the computation







- Solution 1: Adding intermediate stresses point where stresses are calculated (Dyka and Ingel 1995 ; Dyka et al. 1997 ; Randles and Libersky 2000)

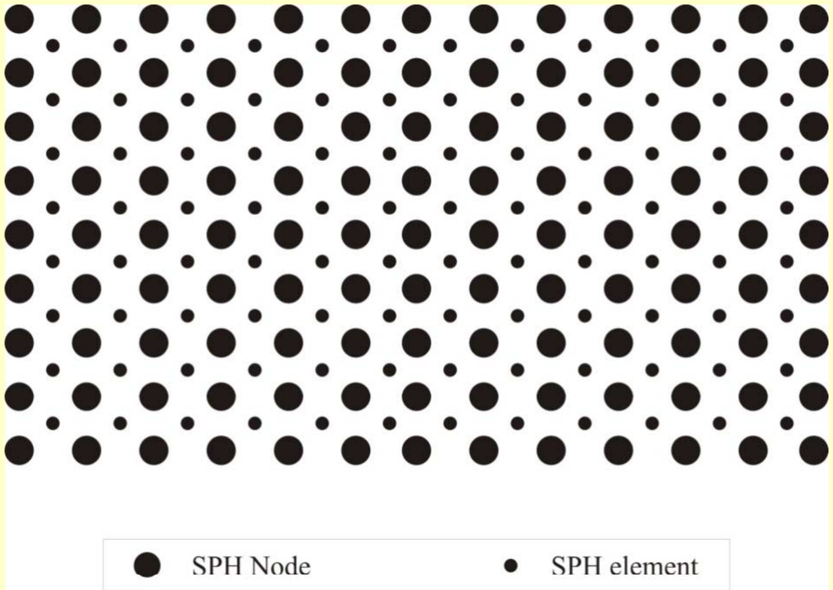
- Solution 2 : The **Runge-Kutta Taylor-SPH model**

Numerical model – The Runge-Kutta Taylor-SPH model

- Based on the mathematical model described in the first part

- Discretization in time following the two-step Taylor-Galerkin method

- Discretization in space with the SPH method using the following particles grid



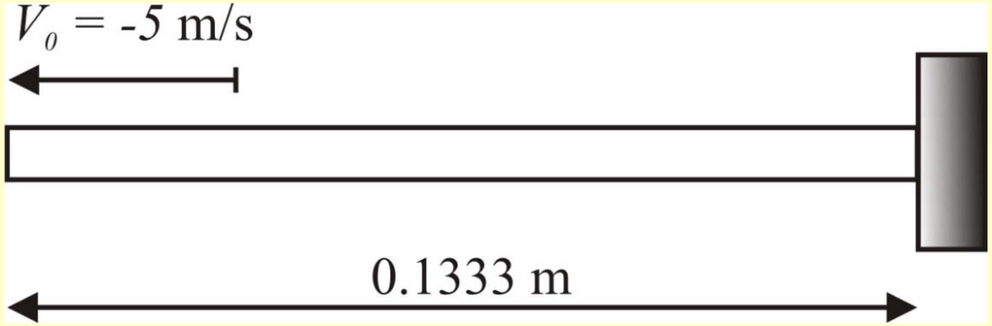
- We calculate ϕ^{n} and ϕ^{n+1} in the SPH nodes

- We calculate $\phi^{n+\frac{1}{2}}$ in the SPH elements

- Implemented in Fortran 90 with output files for a visualization with GID software

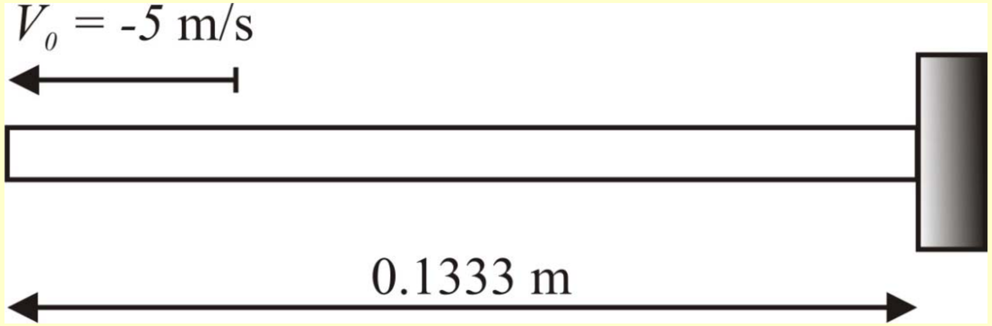
Example 1 – 1D elastic bar

- Bar under tension➔ test of stability

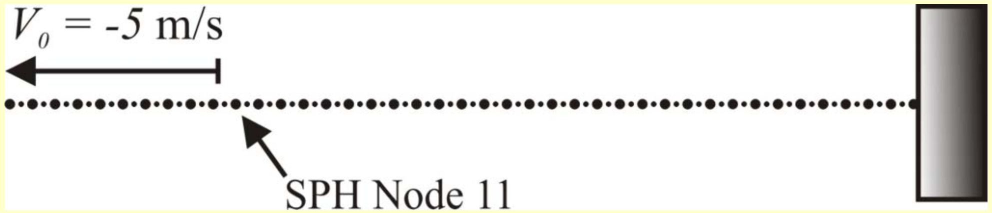


Example 1 – 1D elastic bar

- Bar under tension➔ test of stability



- Comparison with the results given by Dyka and Ingel (1995)

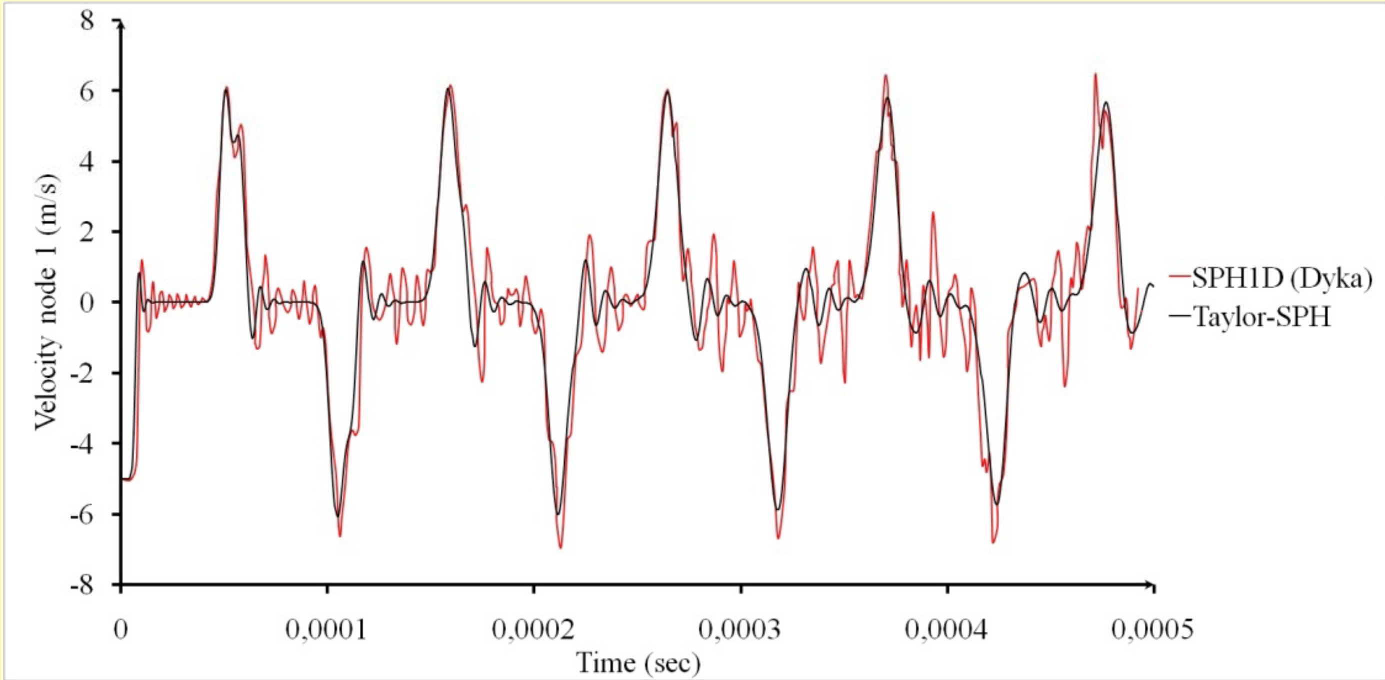


$$\sigma_{node\,1} = 0 \qquad t \in [0, \infty]$$

$$v_{node\,41} = 0 \qquad t \in [0, \infty]$$

Example 1 – 1D elastic bar

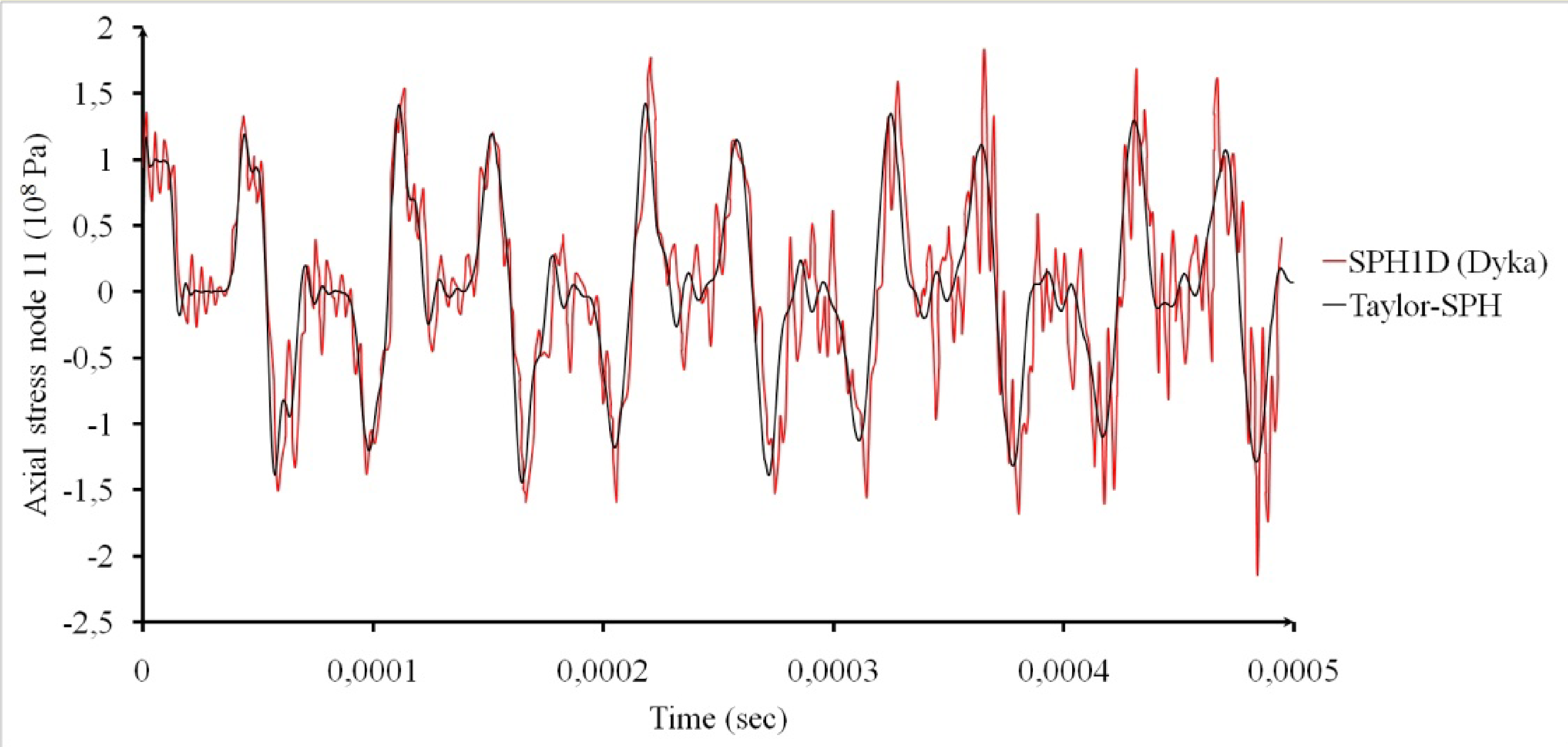
- Velocity at the left extremity of the bar



- **Similar amplitude and less oscillation**

Example 1 – 1D elastic bar

- Axial stress history at the middle of the bar

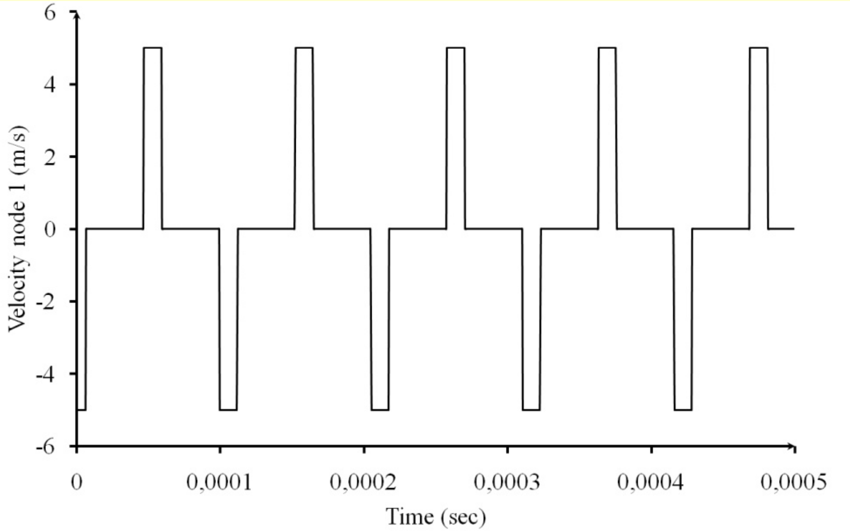


- **<u>Similar amplitude and less oscillation</u>**

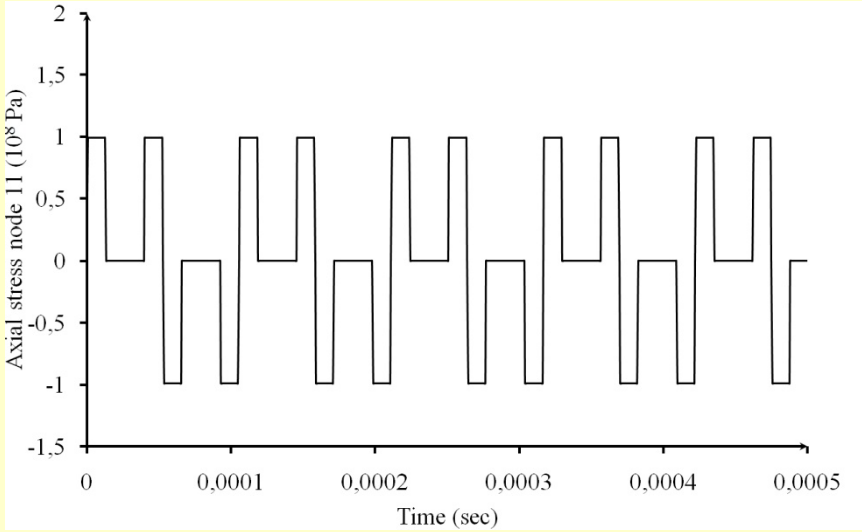
Example 1 – 1D elastic bar

- Same example with a Courant's number equals to 1

Velocity



Stress

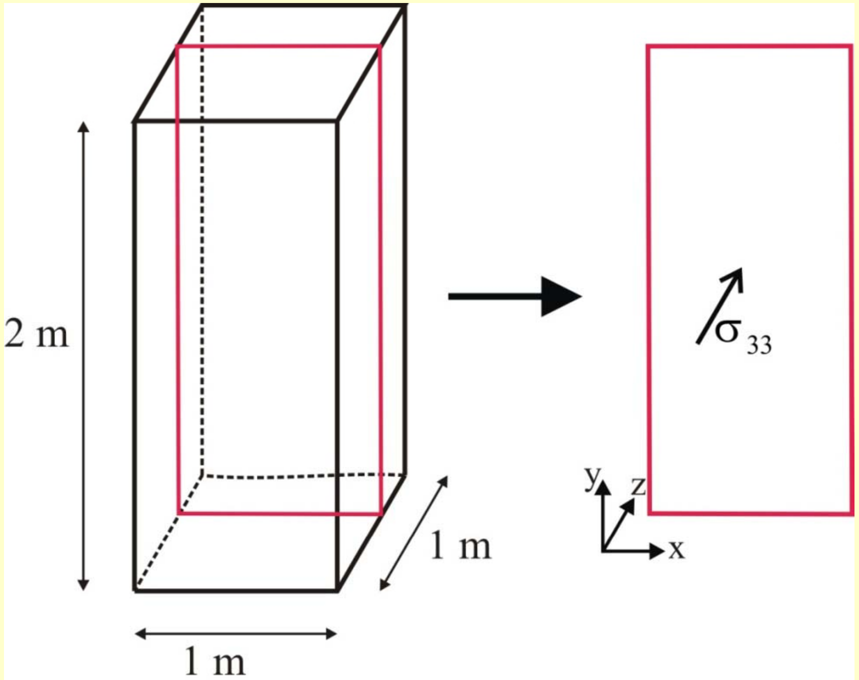


- **<u>Solution given by the Runge-Kutta Taylor-SPH match with the analytical solution</u>**

- **<u>Our algorithm allows avoiding the SPH tensile instability</u>**

Example 2 – Strain localization in 2D soil sample

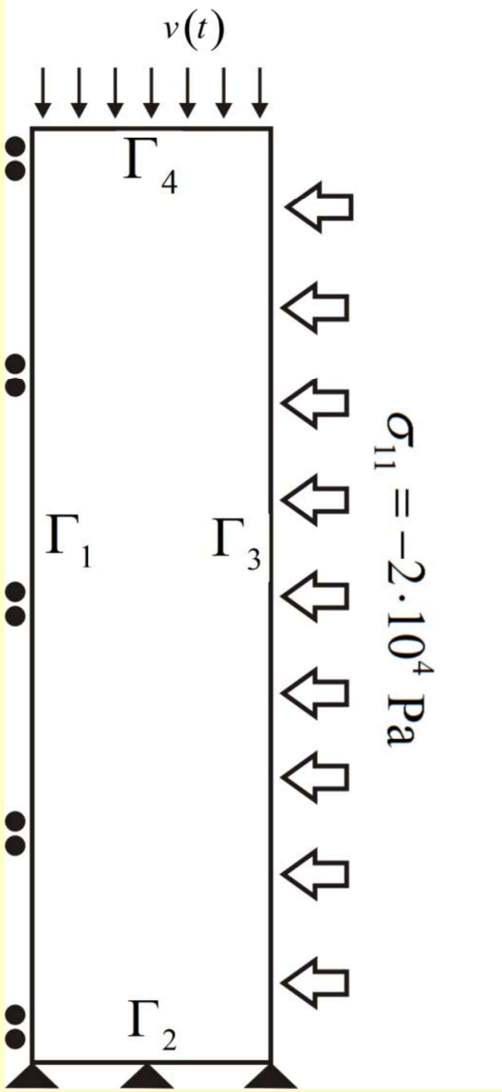
- 2D soil sample (plane strain condition)



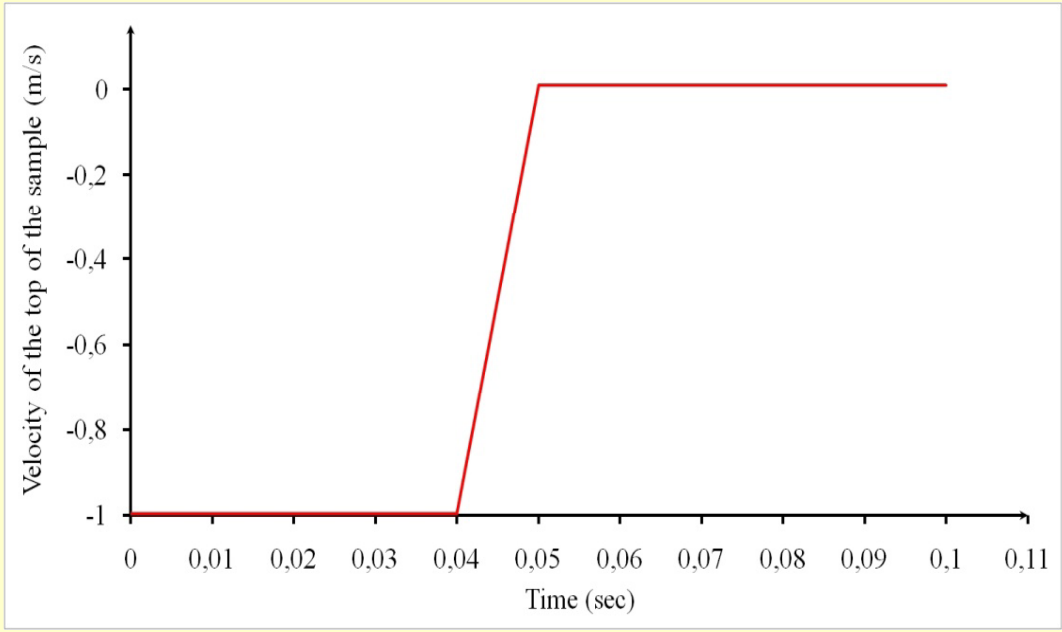
- Viscoplastic material (Cam-Clay) with constant initial stress state

$$\sigma_{11_0} = \sigma_{22_0} = \sigma_{12_0} = -2 \cdot 10^{-4} \ \text{Pa}$$

Example 2 – Strain localization in 2D soil sample

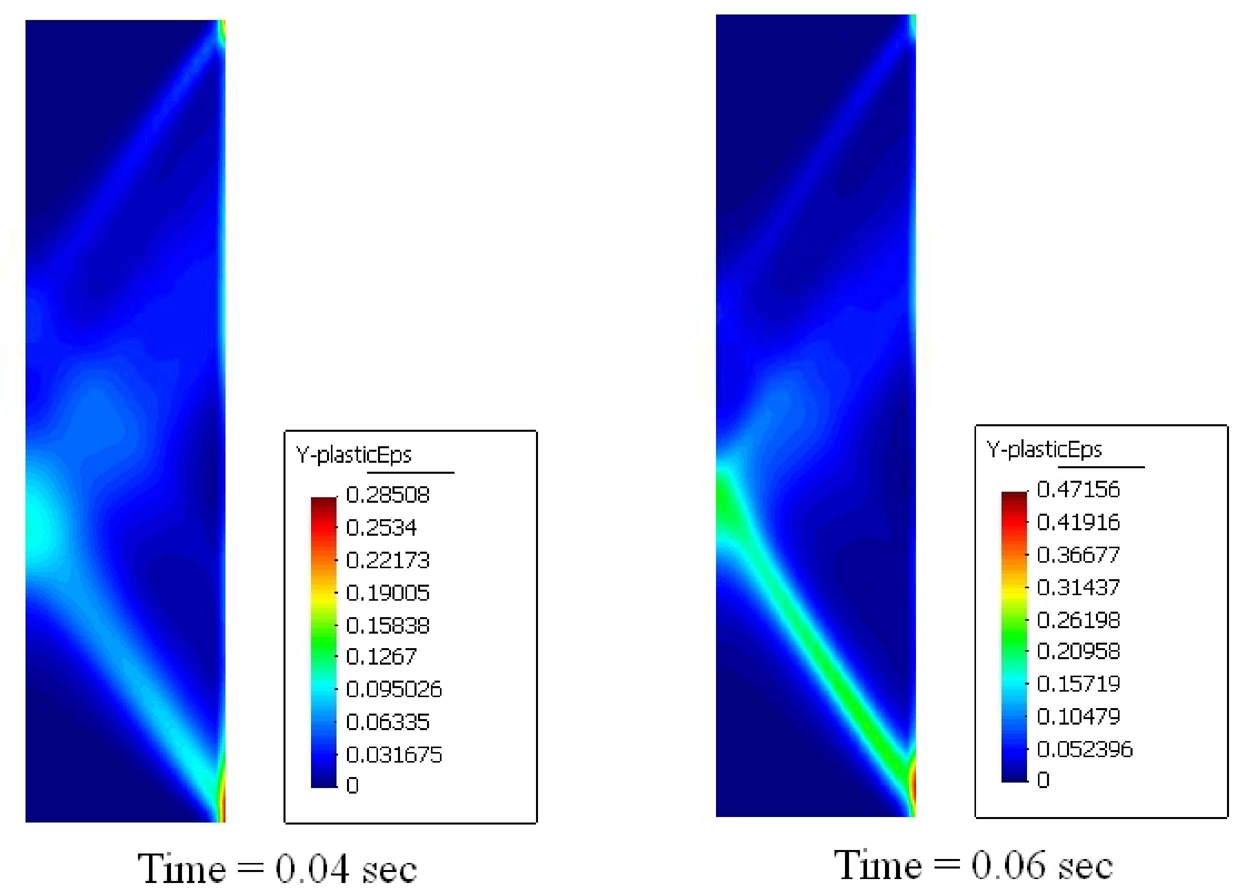


Compression shock wave (negative velocity)



Example 2 – Strain localization in 2D soil sample

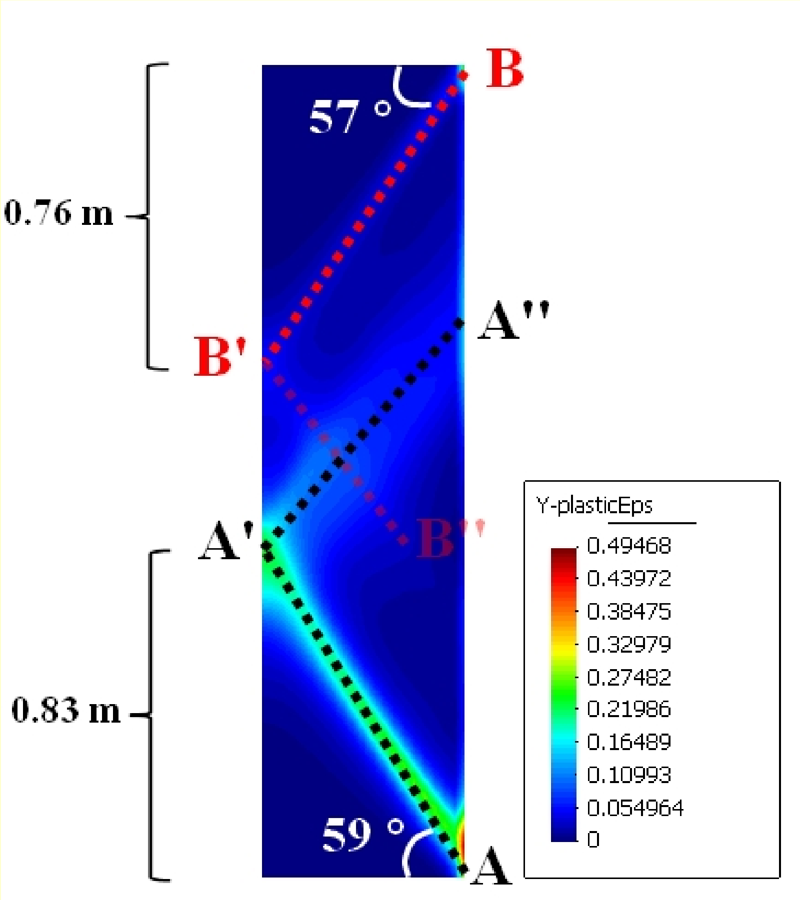
- Viscoplastic strains



- **<u>Localization in shear bands</u>**

Example 2 – Strain localization in 2D soil sample

- Orientation of the shear bands *(Desrues 1987)*



$$\theta_{shear\ band} = \frac{\pi}{4} + \frac{\phi}{2}$$

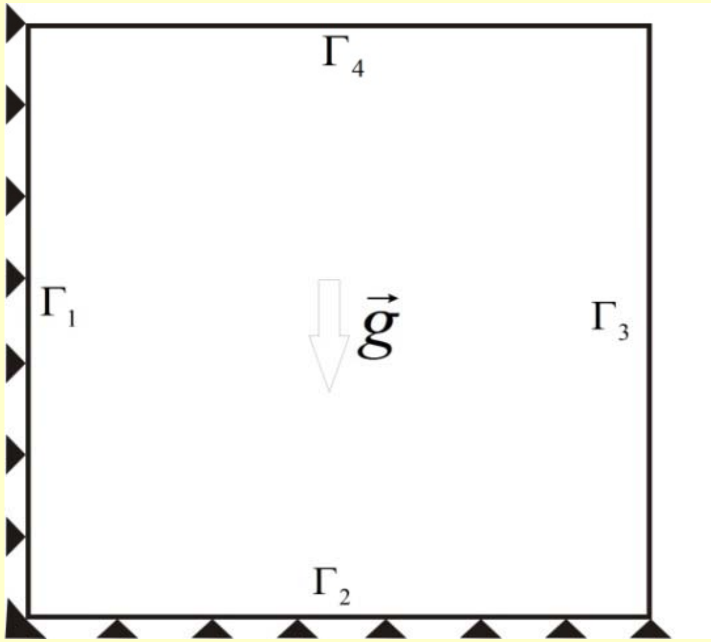
- **<u>Good prediction of the inclination of the shear bands</u>**

Example 3 – Vertical slope stability

- Vertical slope is 10 meters high– Von Mises material without softening
- Shear stress reduction method to find the stability factor of the vertical slope
- Initial size of the yield surface $\sigma_0 = 200000 \text{ Pa}$ which decreases along time until failure of the vertical slope

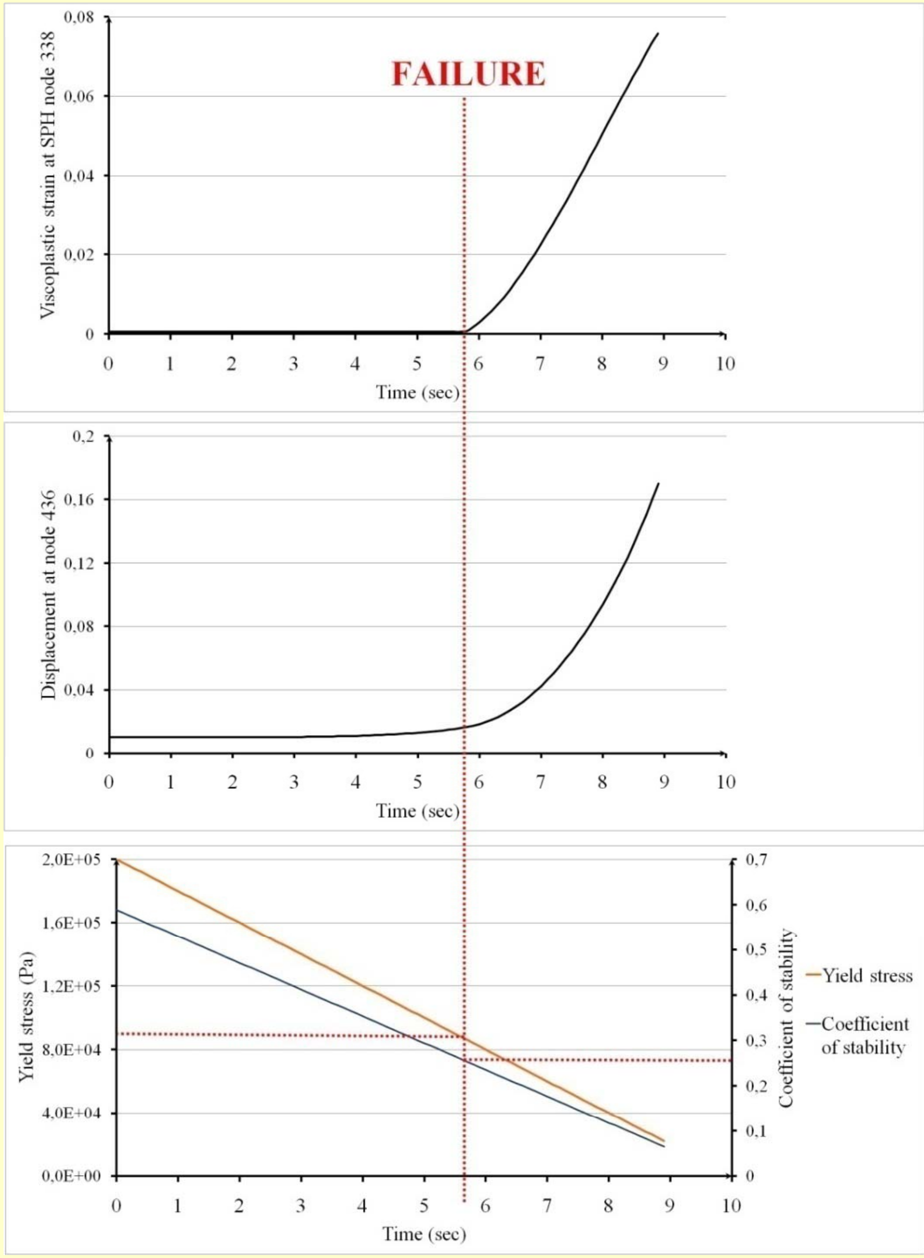
- Stability factor

$$FS = \frac{\sigma_{0,\,failure}}{\sqrt{3}\gamma H}$$



Example 3 – Vertical slope stability

- Determination of the vertical slope failure



$$\sigma_{0,\,failure} = 86024 \text{ Pa}$$

$$FS_{Taylor-SPH} = 0.253$$

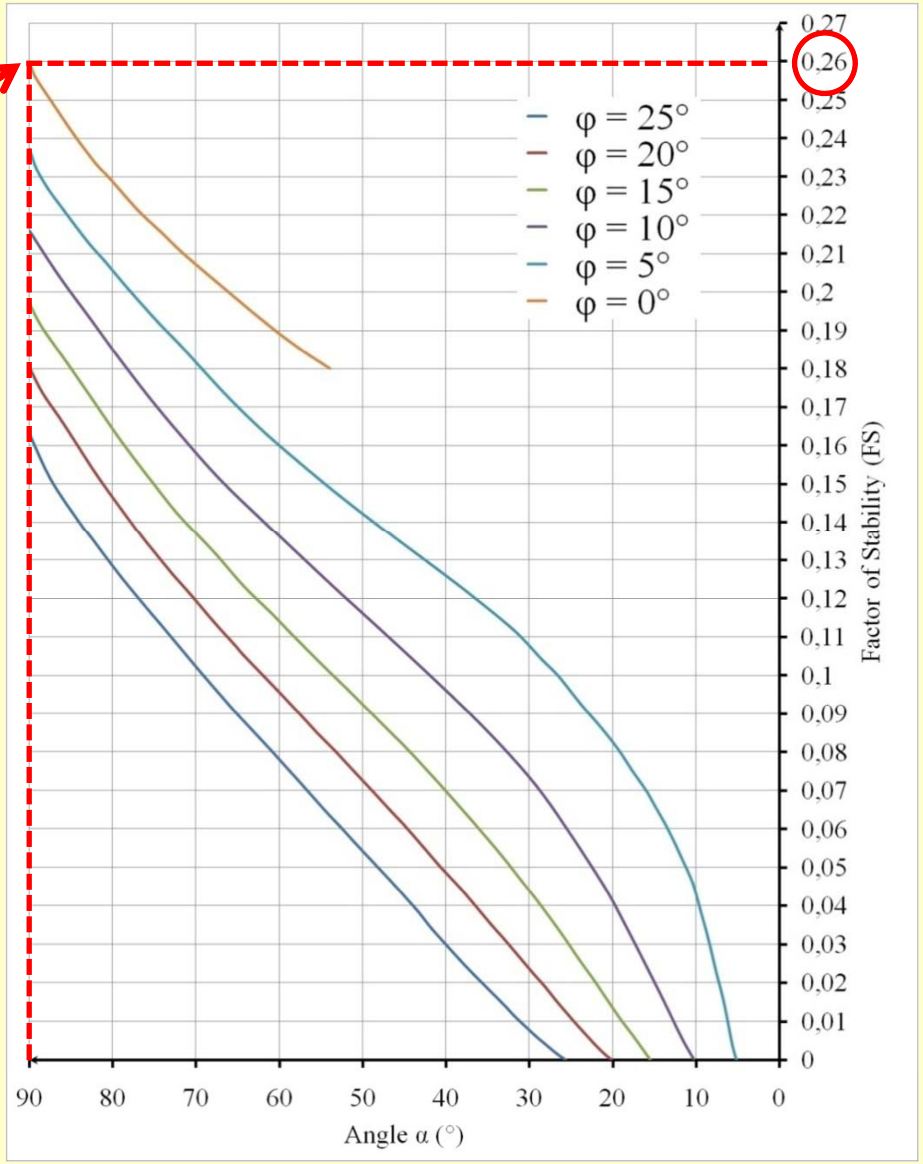
Example 3 – Vertical slope stability

- Taylor-s abacus

$$FS_{Taylor\ abacus} = 0.26$$

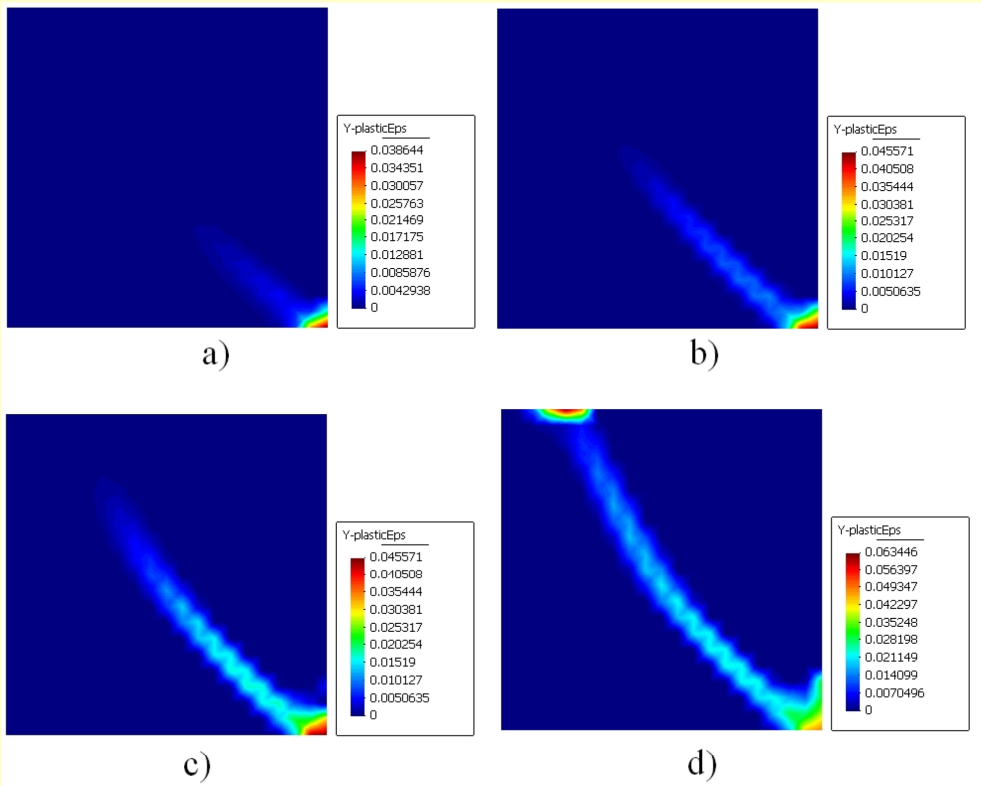
$$FS_{Taylor-SPH} = 0.253$$

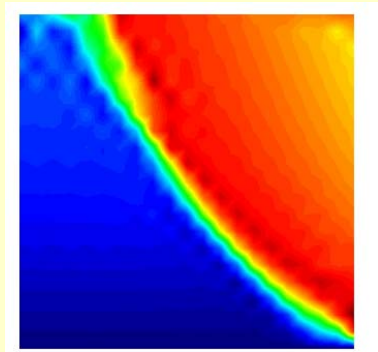
- **<u>Accurate prediction of the factor of stability</u>**

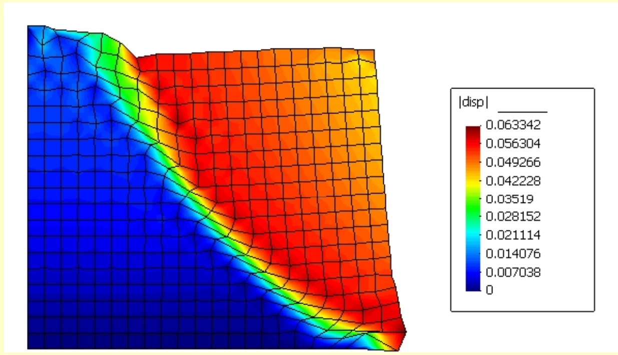


Example 3 – Vertical slope stability

- Same example of the vertical slope but with softening







- **<u>Localization of the viscoplastic strains along a well defined shear band and failure of the vertical slope</u>**

Contents

- Introduction
- SPH (I) Basic Model
- SPH (II) Large deformation
- SPH (III) Coupled v-p_w
- SPH (IV) Depth integrated models h-v-p_w

Requiered modification of the model

- Large deformations ➔ Updating of the particle position at each step

    ➔Modification of the mathematical model equations

    ➔ Problem of Hourglass deformation

    ➔ Updating of the smoothing length

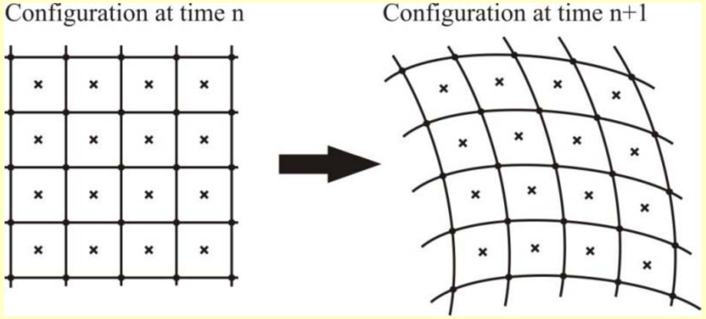
    ➔ Boundary condition and free surface

Mathematical model equations

- Large deformations ➔ Updating of the particle position at each step

$$\boldsymbol{x}^{n+1}_{\text{SPH Node}} = \boldsymbol{x}^{n}_{\text{SPH Node}} + \Delta t \cdot \boldsymbol{v}^{n+1/2}_{\text{SPH Node}}$$

$$\boldsymbol{x}^{n+1}_{\text{SPH el}} = \frac{1}{4}\sum_{j=1}^{4} \boldsymbol{x}^{n+1}_{\text{SPH Node } j}$$



- Large deformations ➔ Stress rate $\dot{\sigma}_{ij}$ replaced by the Jaumann stress rate $\dot{\hat{\sigma}}_{ij}$

➔ takes into account the rotation in the solid

$$\dot{\hat{\sigma}}_{ij} = \dot{\sigma}_{ij} - \sigma_{ik}\dot{\omega}_{jk} - \sigma_{kj}\dot{\omega}_{ik}$$

Where the spin rate tensor is

$$\dot{\omega}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

Mathematical model equation

- Hyperbolic system of equations in the large deformation theory

$$\frac{\partial}{\partial t}\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\\ \sigma_{33}\\ v_1\\ v_2\end{pmatrix}-\underbrace{\begin{pmatrix}\sigma_{1k}\dot{\omega}_{1k}+\sigma_{k1}\dot{\omega}_{1k}\\ \sigma_{2k}\dot{\omega}_{2k}+\sigma_{k2}\dot{\omega}_{2k}\\ \sigma_{1k}\dot{\omega}_{2k}+\sigma_{k2}\dot{\omega}_{1k}\\ 0\\ 0\\ 0\end{pmatrix}}_{\text{Jaumann stress rate}}-\frac{\partial}{\partial x_1}\begin{pmatrix}D_{11}v_1\\ D_{12}v_1\\ D_{33}v_2\\ D_{41}v_1\\ \sigma_{11}/\rho\\ \sigma_{12}/\rho\end{pmatrix}-\frac{\partial}{\partial x_2}\begin{pmatrix}D_{12}v_2\\ D_{22}v_2\\ D_{33}v_1\\ D_{42}v_2\\ \sigma_{12}/\rho\\ \sigma_{22}/\rho\end{pmatrix}+\begin{pmatrix}D_{11}\dot{\varepsilon}_{11}^{vp}+D_{12}\dot{\varepsilon}_{22}^{vp}\\ D_{12}\dot{\varepsilon}_{11}^{vp}+D_{22}\dot{\varepsilon}_{22}^{vp}\\ D_{33}\dot{\varepsilon}_{12}^{vp}\\ D_{41}\dot{\varepsilon}_{11}^{vp}+D_{42}\dot{\varepsilon}_{22}^{vp}+D_{44}\dot{\varepsilon}_{33}^{vp}\\ b_1\\ b_2\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\\ 0\\ 0\\ 0\end{pmatrix}$$

Boundary conditions

- Conditions which can be directly applied on the boundary:
  - $v_x = 0$ o $v_y = 0$
  - $v_x = v_y = 0$

- Conditions on the free surface
  - $\sigma_n = 0$ y $\tau = 0$

- Updating the SPH nodes position

  Which nodes define the free surface ?

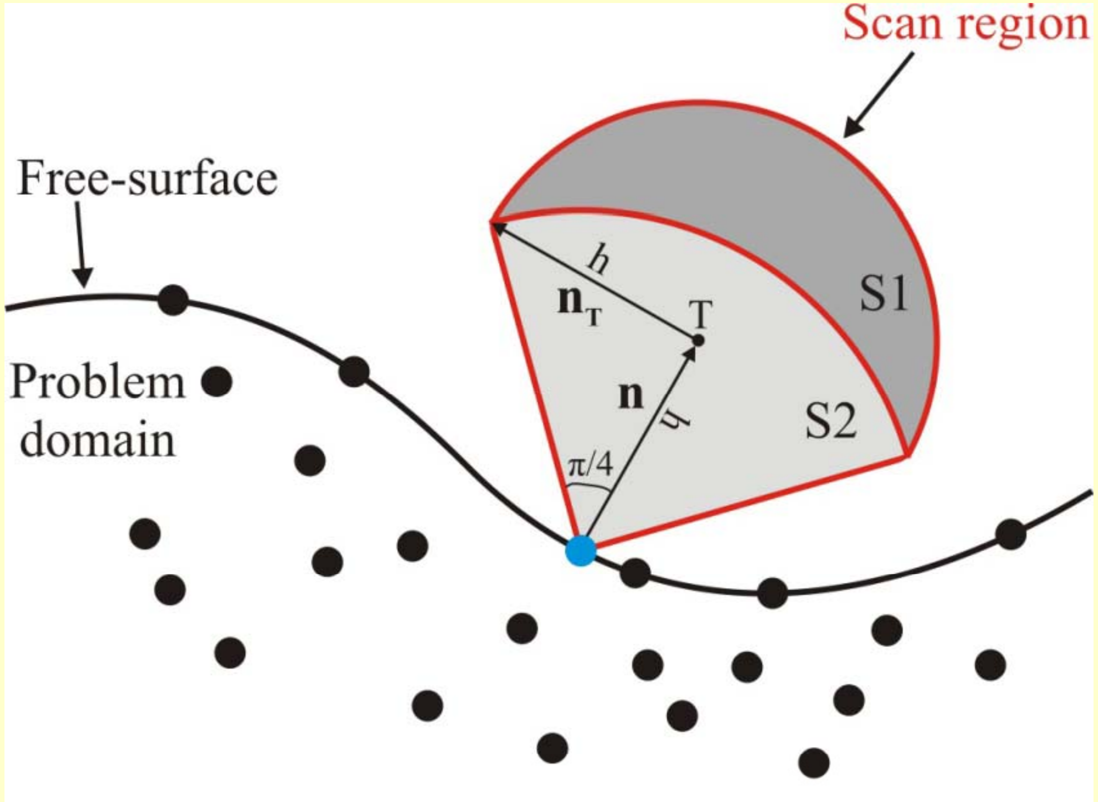
  What is the normal free surface ?

➔Algorithm of free-surface detection and calculation of the normal to the free-surface

- Modification of the algorithm proposed by Marrone and al. (2010)

TSPH d2 Boundary conditions:

Detection of boundary nodes (Marrone et al 2010)

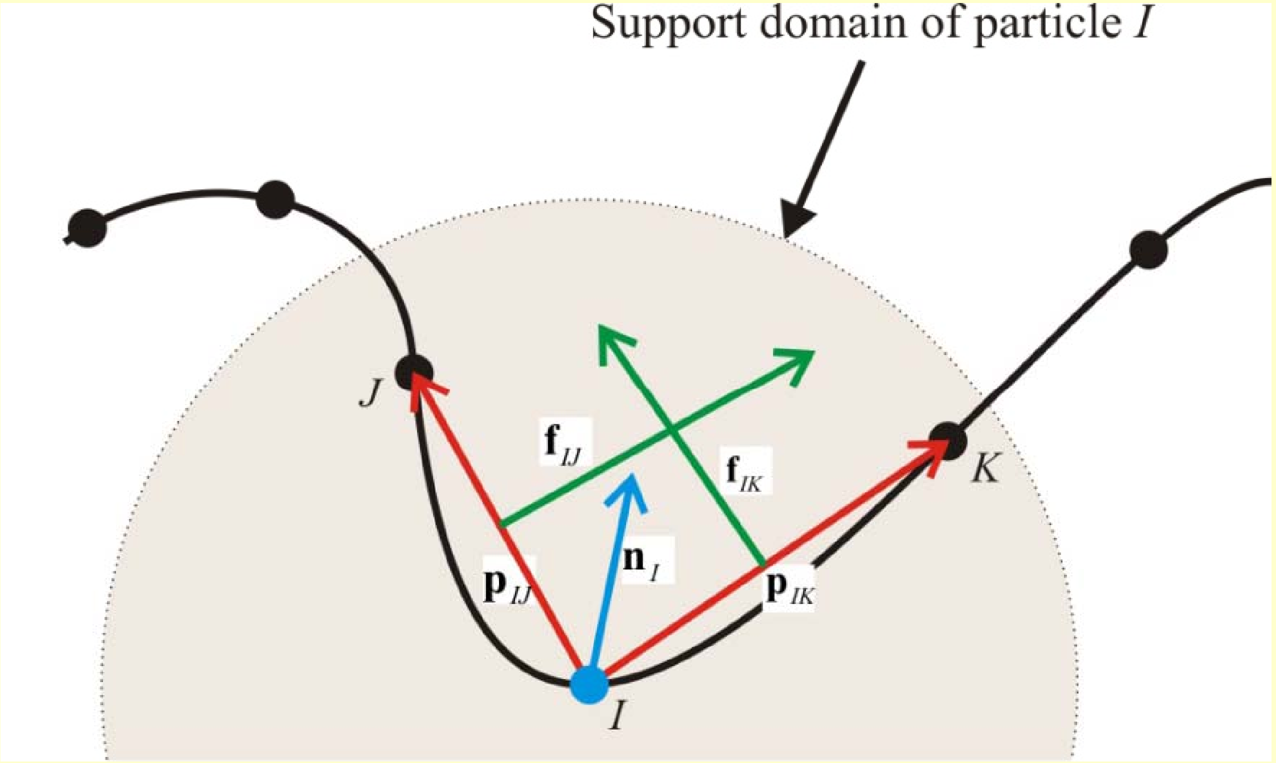


If any particle is inside the scan region of the particle, then the particle does not belong to the free-surface

TSPH d2 Boundary conditions:

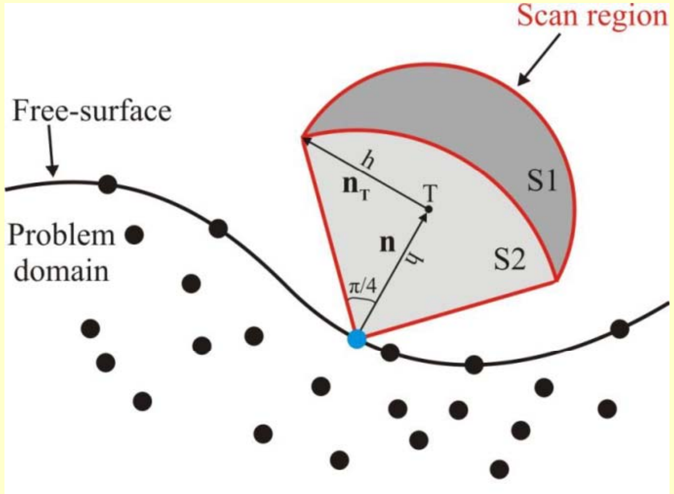
Computation of unit normals

$$\boldsymbol{n}_I = \frac{1}{N_P}\sum_{J=1}^{N_P} \boldsymbol{f}_{IJ}$$



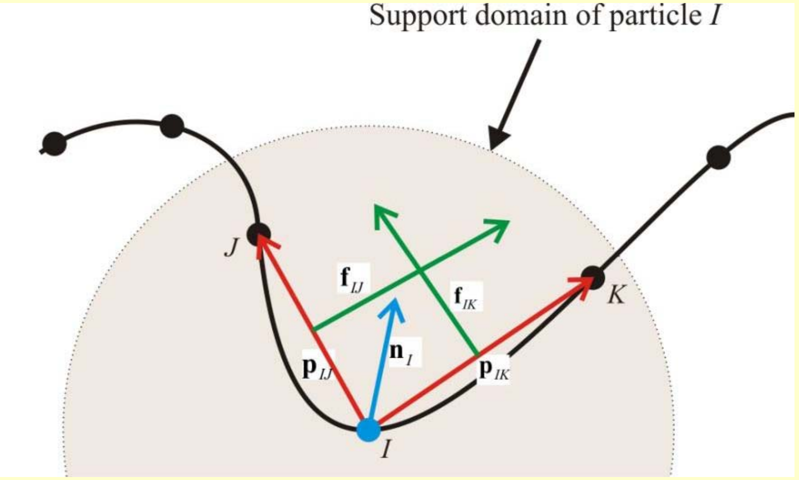
Boundary conditions

- Algorithm:
  - •First step: Free-surface detection (similar to the algorithm of Marrone et al 2010)



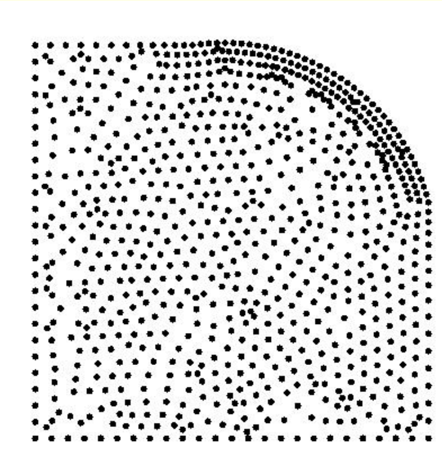
  - • Second step: Calculation of the normal

$$\boldsymbol{n}_I = \frac{1}{N_P} \sum_{J=1}^{N_P} \boldsymbol{f}_{IJ}$$

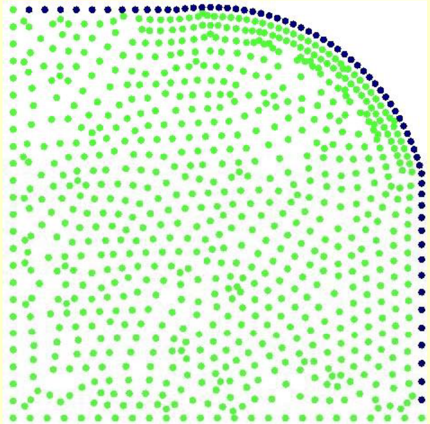


Boundary conditions

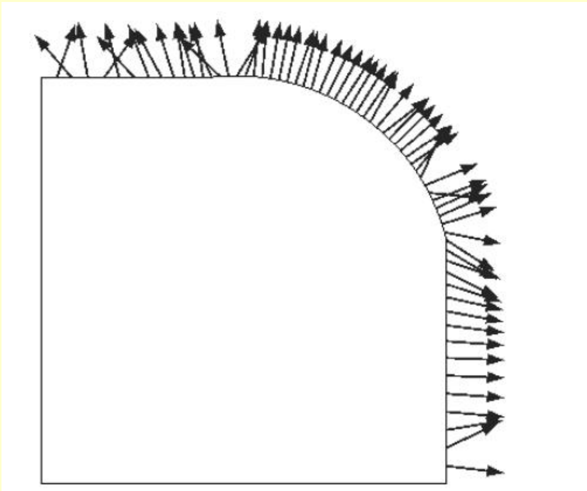
- Example:



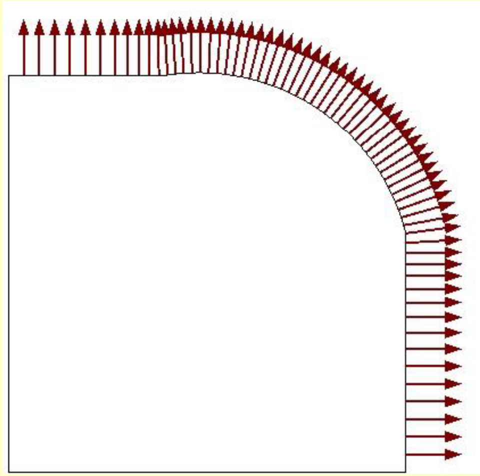
First step



Second step



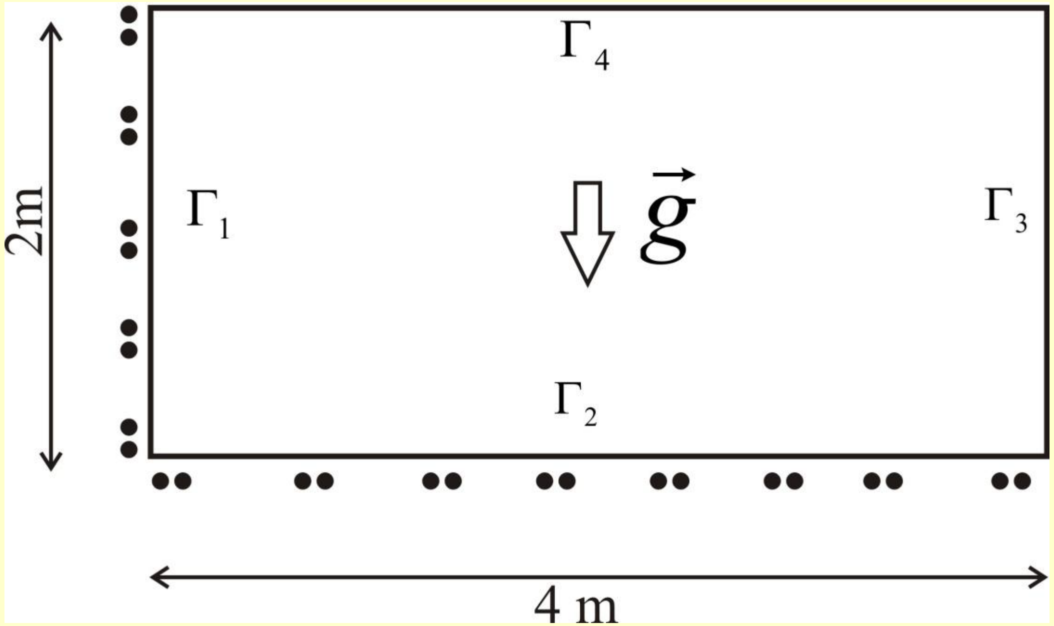
With algorithm proposed by *Marrone et al*



Proposed algorithm

Example 1 – Failure of an extremely low cohesive vertical slope under gravity

- Boundary conditions of the vertical slope



Example 1 – Failure of an extremely low cohesive vertical slope under gravity

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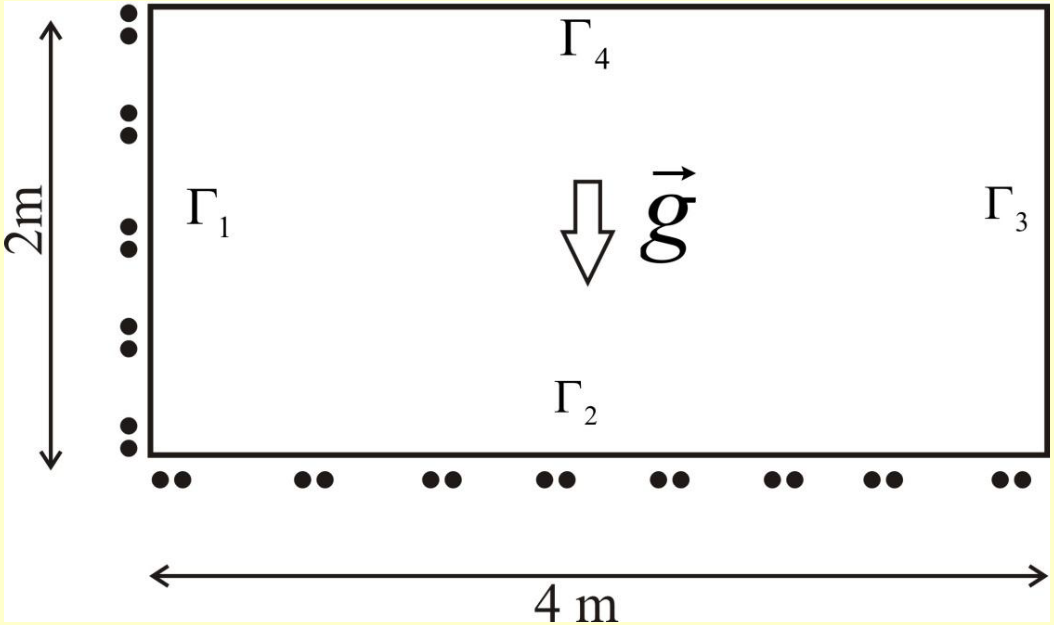
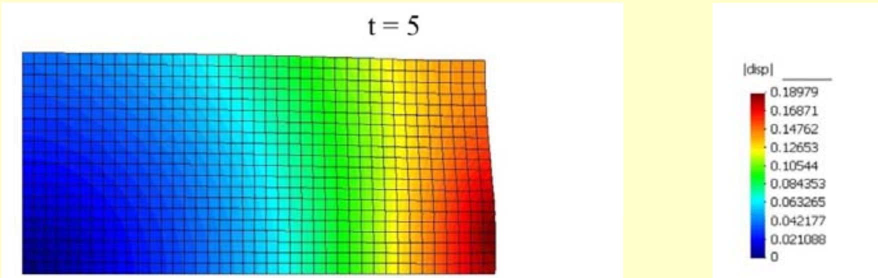
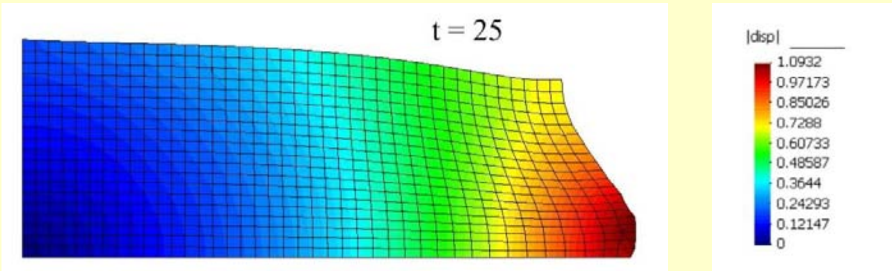


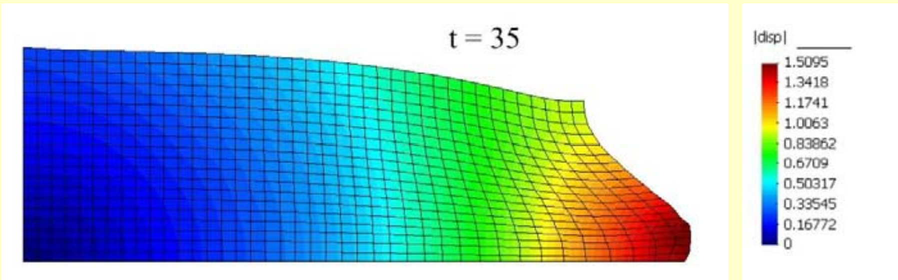
- Viscoplastic material – Von Mises without softening and with cohesion $Y_0 = 2 \cdot 10^3$ Pa

Example 1 – Failure of an extremely low cohesive vertical slope under gravity

- Displacements







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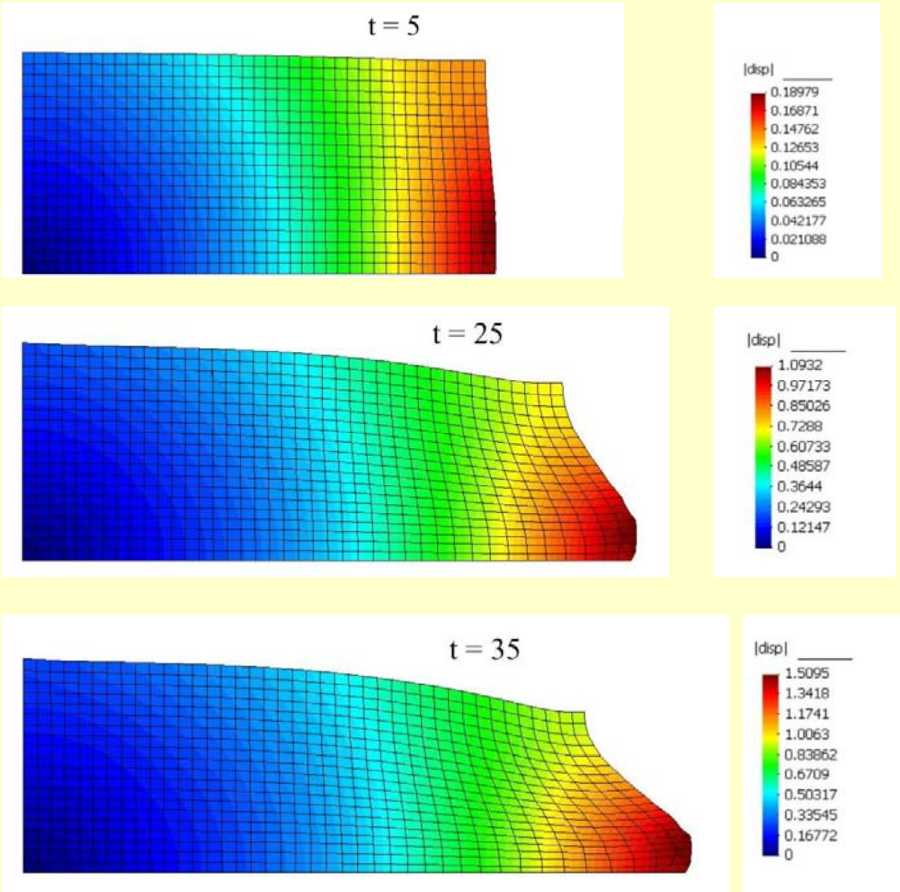
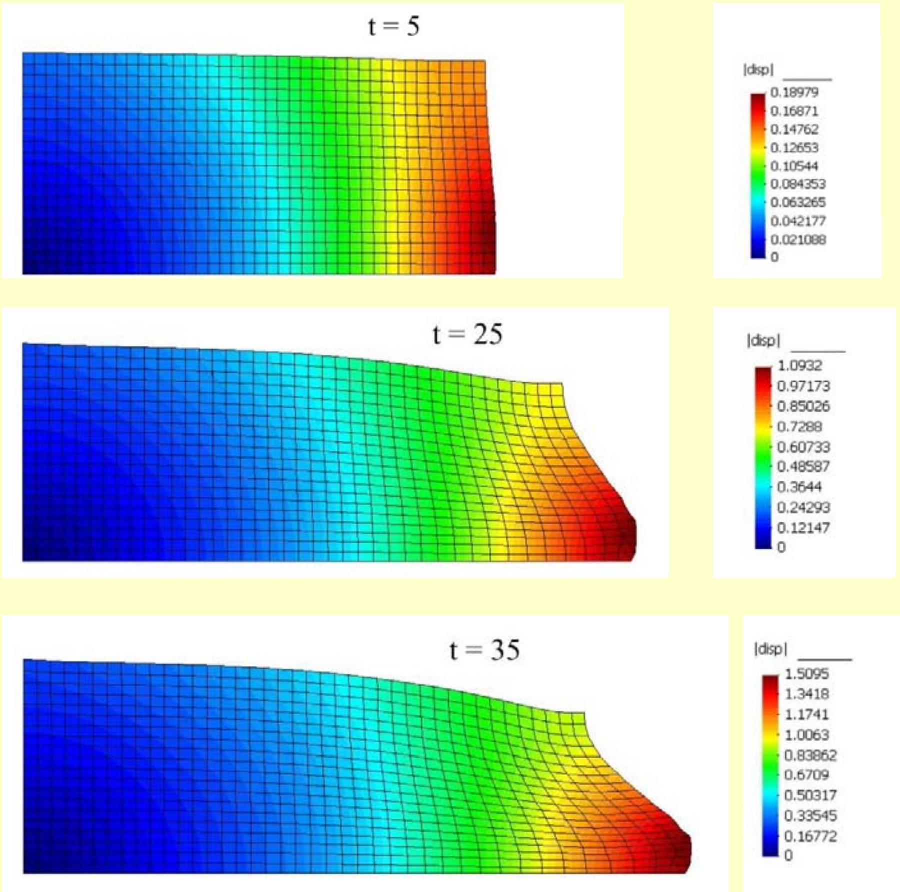


- Large deformation problem
  - Viscoplastic deformations 150 %

Example 1 – Failure of an extremely low cohesive vertical slope under gravity

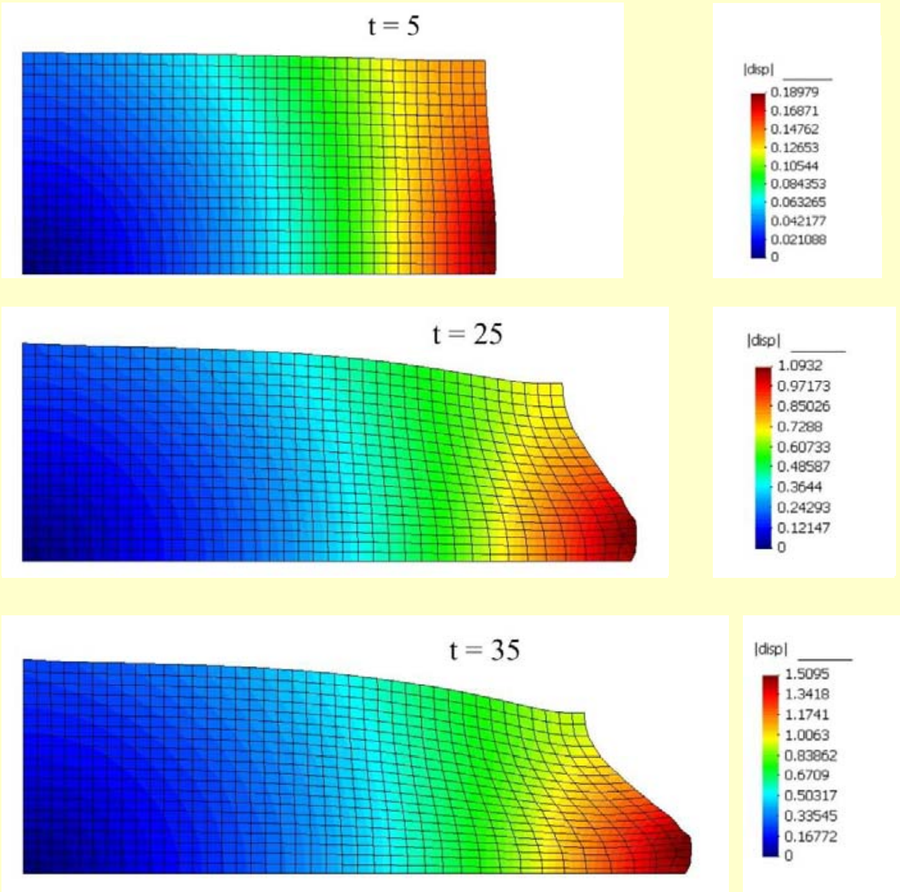
- Displacements



- Large deformation problem
  - Viscoplastic deformations 150 %
  - Updating of the SPH nodes

Example 1 – Failure of an extremely low cohesive vertical slope under gravity

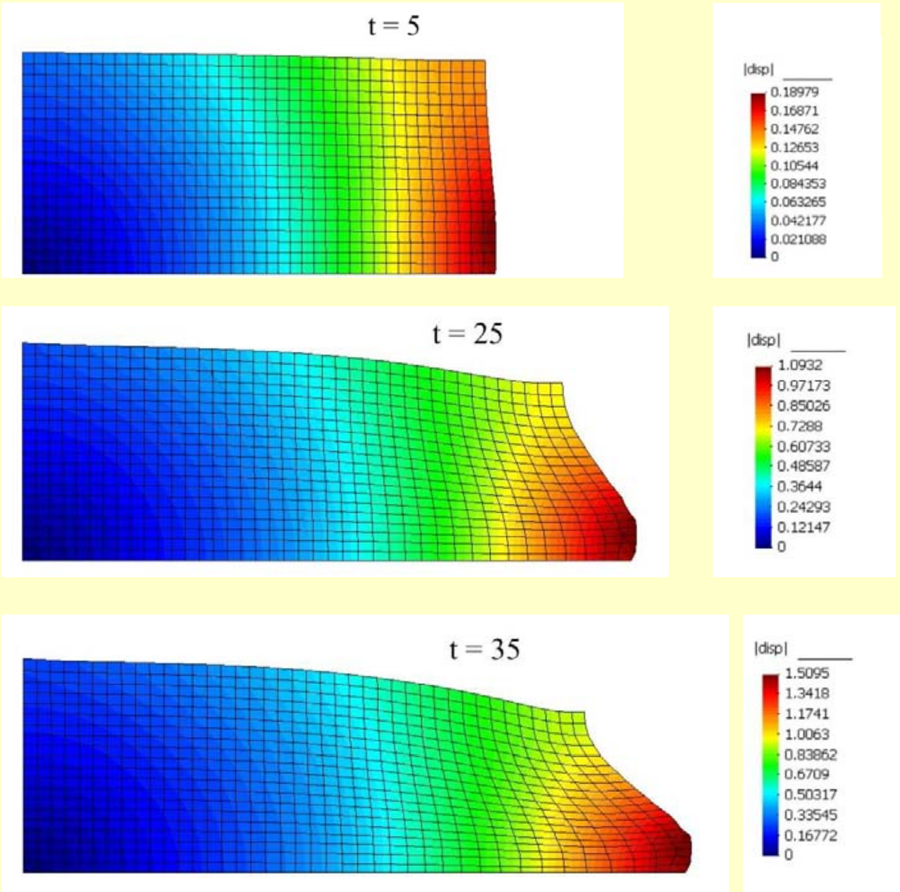
- Displacements



- Large deformation problem
  - Viscoplastic deformations 150 %
  - Updating of the SPH nodes
  - Rotation inside the solid

Example 1 – Failure of an extremely low cohesive vertical slope under gravity

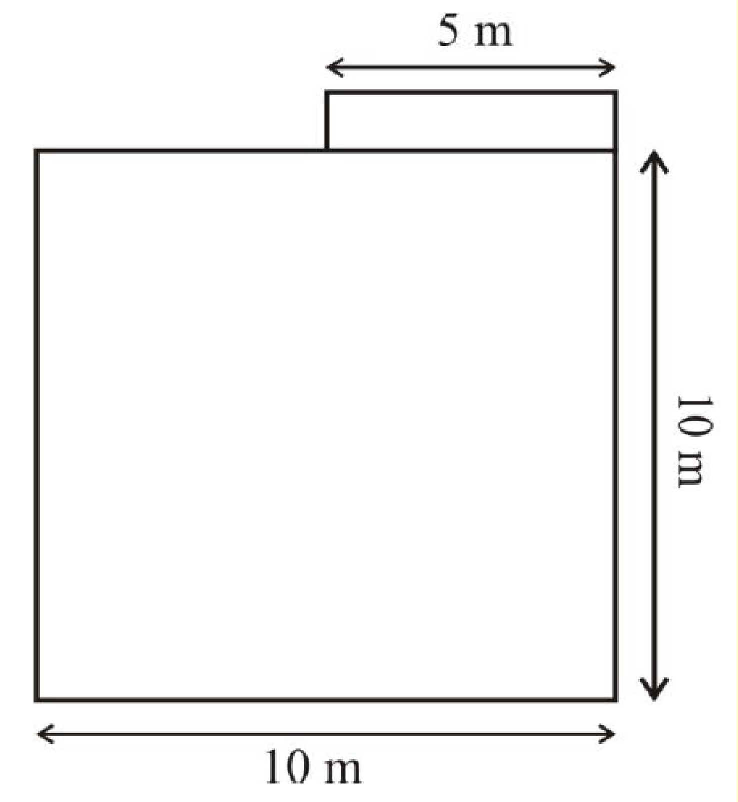
- Displacements

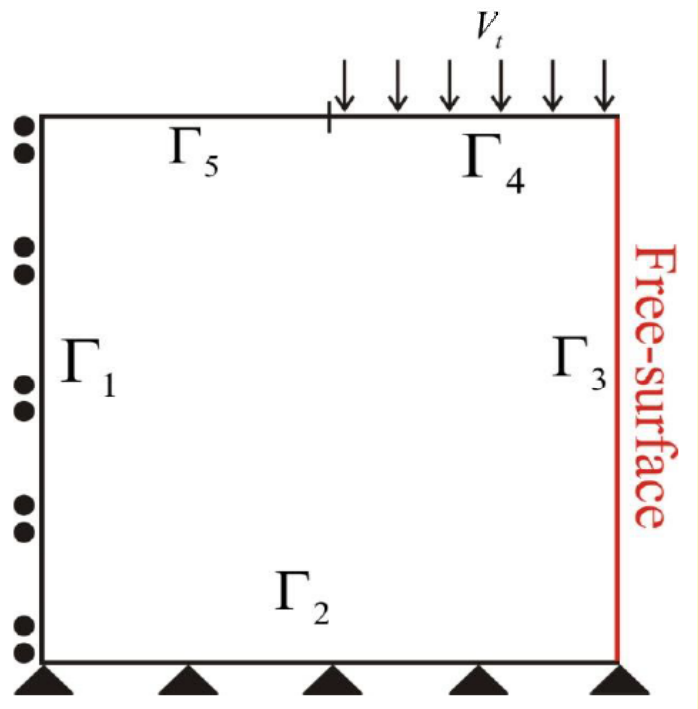


- Large deformation problem
  - Viscoplastic deformations 150 %
  - Updating of the SPH nodes
  - Rotation inside the solid
  - Change of the free-surface

Example 2 – Failure of a vertical cut under constant loading

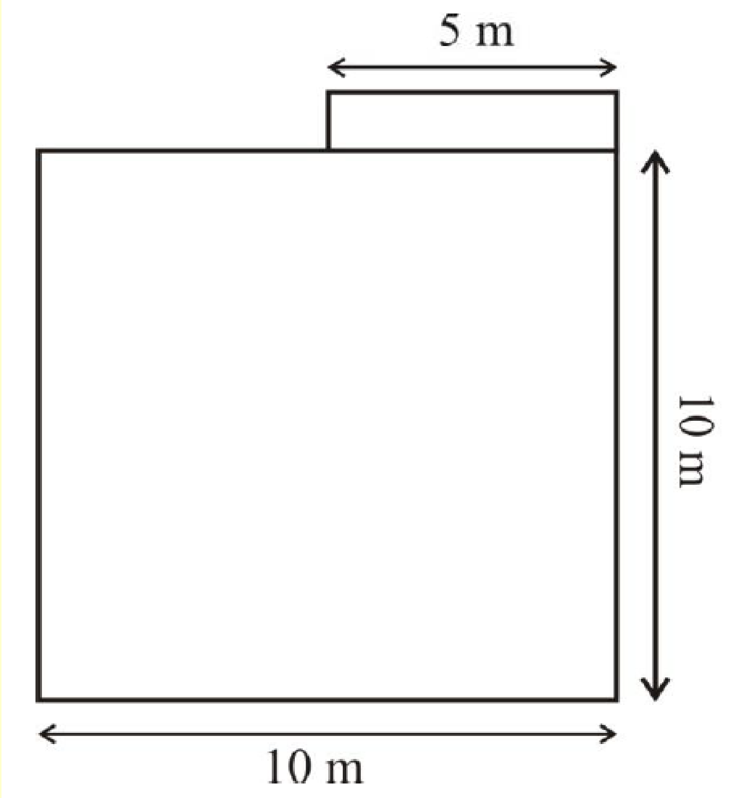
- Boundary conditions of the vertical slope

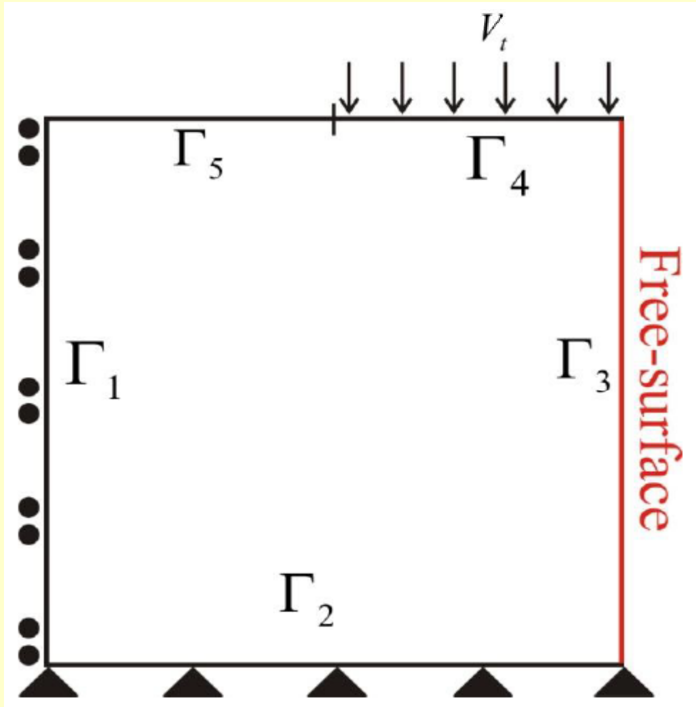




Example 2 – Failure of a vertical cut under constant loading

- Boundary conditions of the vertical slope

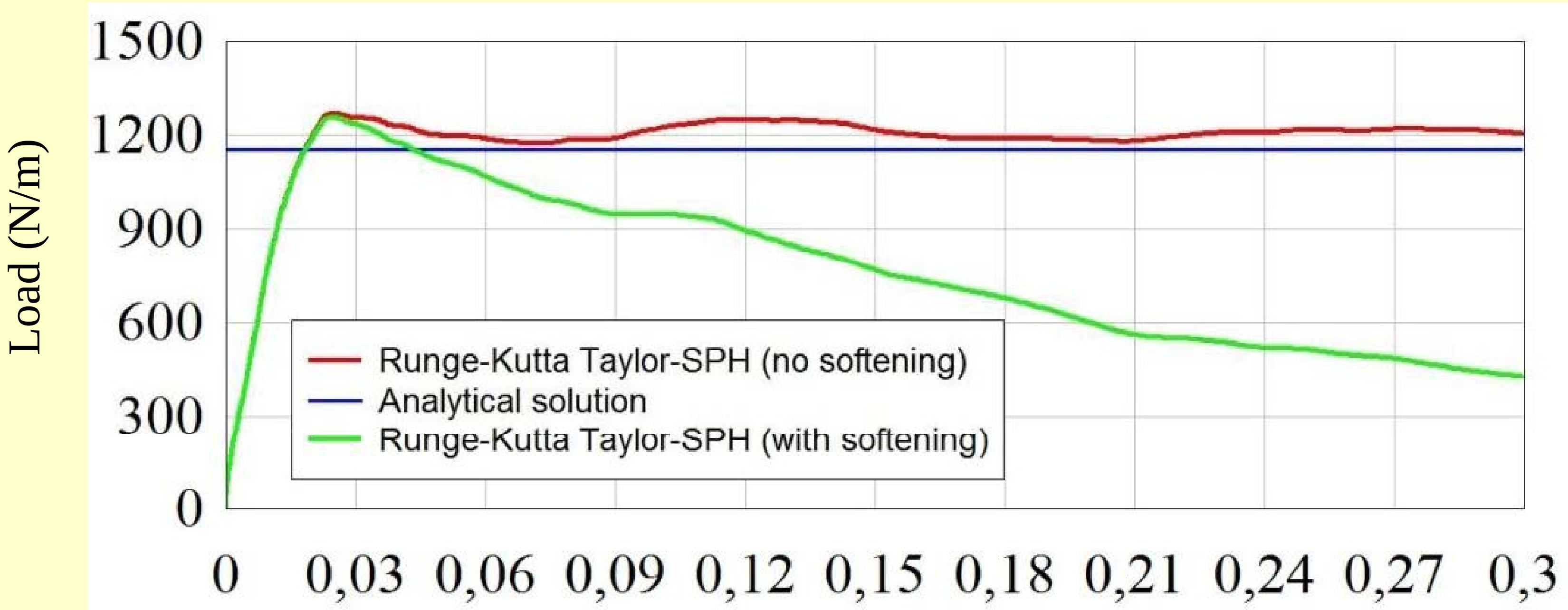




- Viscoplastic material:
  - Von Mises without softening and cohesion $Y_0 = 200\ \text{Pa}$
  - Von Mises with softening $H = -1 \cdot 10^3\ \text{Pa}$ and cohesion $Y_0 = 200\ \text{Pa}$

Example 2 – Failure of a vertical cut under constant loading

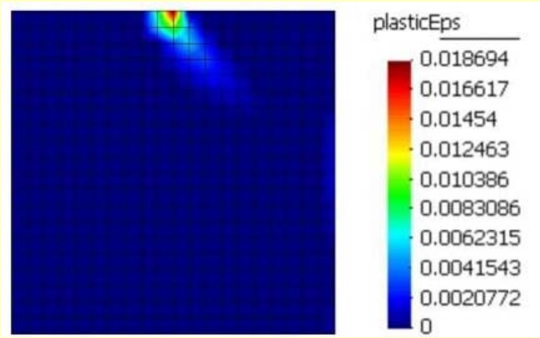
- Analytical solution: Limit load $P_{analitycal} = \frac{2bY_0}{\sqrt{3}}$ ➔ $P_{analitycal} = 1154$ N



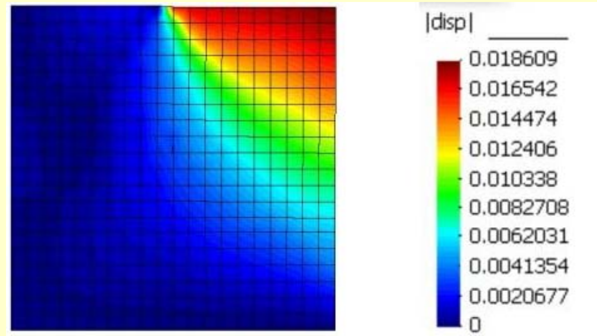
Displacements (m)

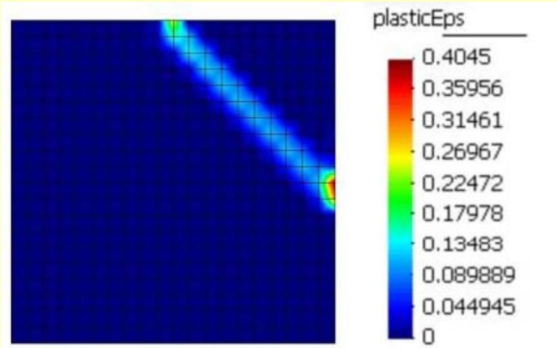
Example 2 – Failure of a vertical cut under constant loading

- Viscoplastic deformations and displacements

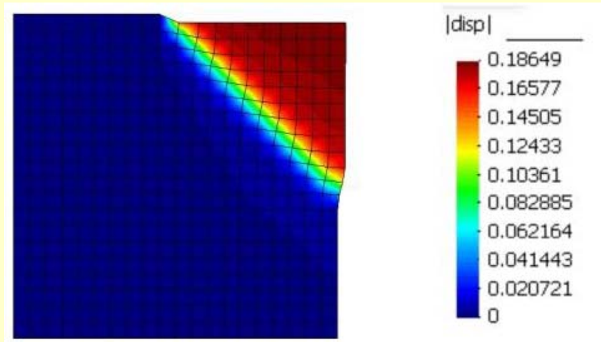


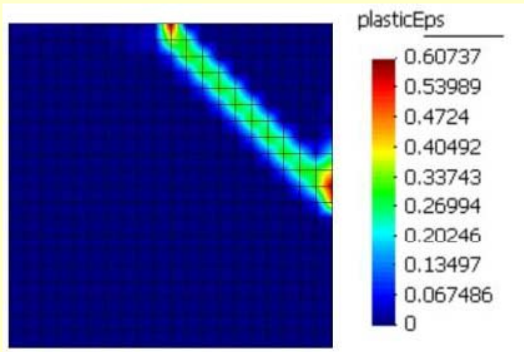
t = 4.5 s



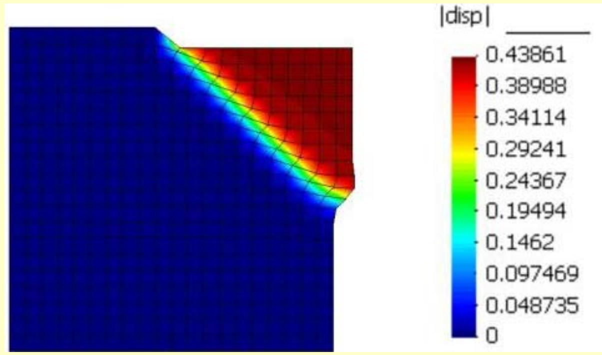


t = 13 s



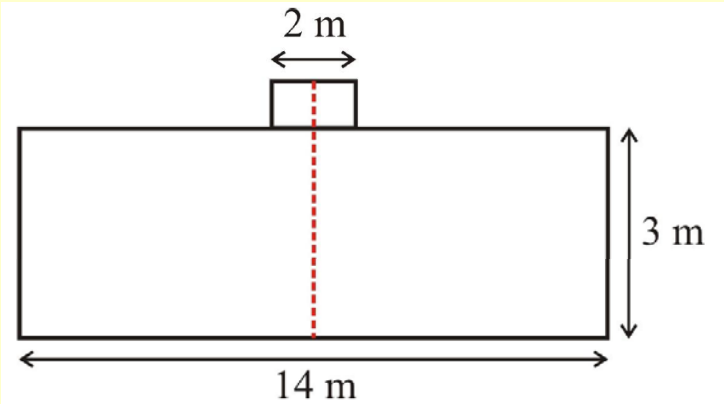


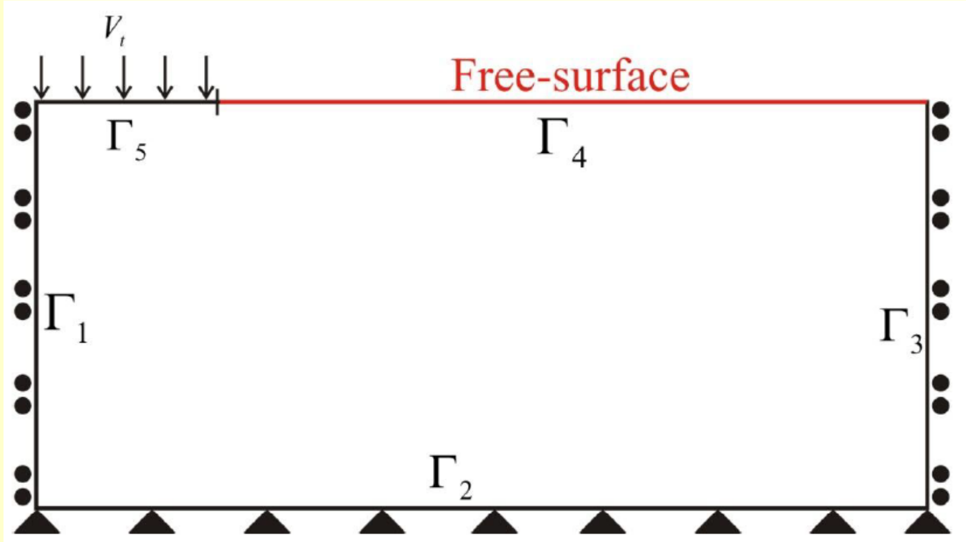
t = 20 s



Example 3 – Failure of a shallow stratum under wide strip footing

- Boundary conditions

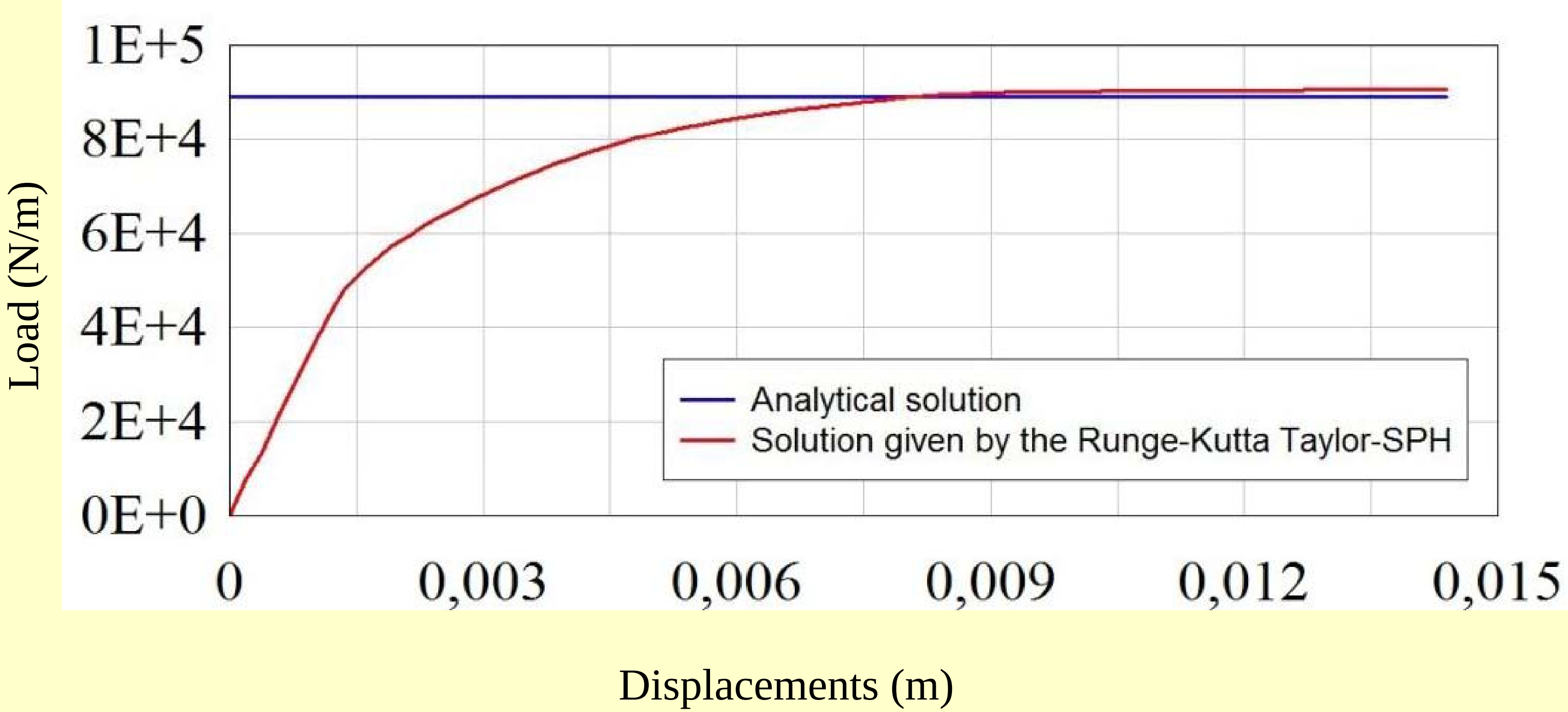




- Viscoplastic material:
  - Von Mises without softening and cohesion $Y_0 = 15000$ Pa
  - Von Mises with softening $H = -1 \cdot 10^6$ Pa and cohesion $Y_0 = 15000$ Pa

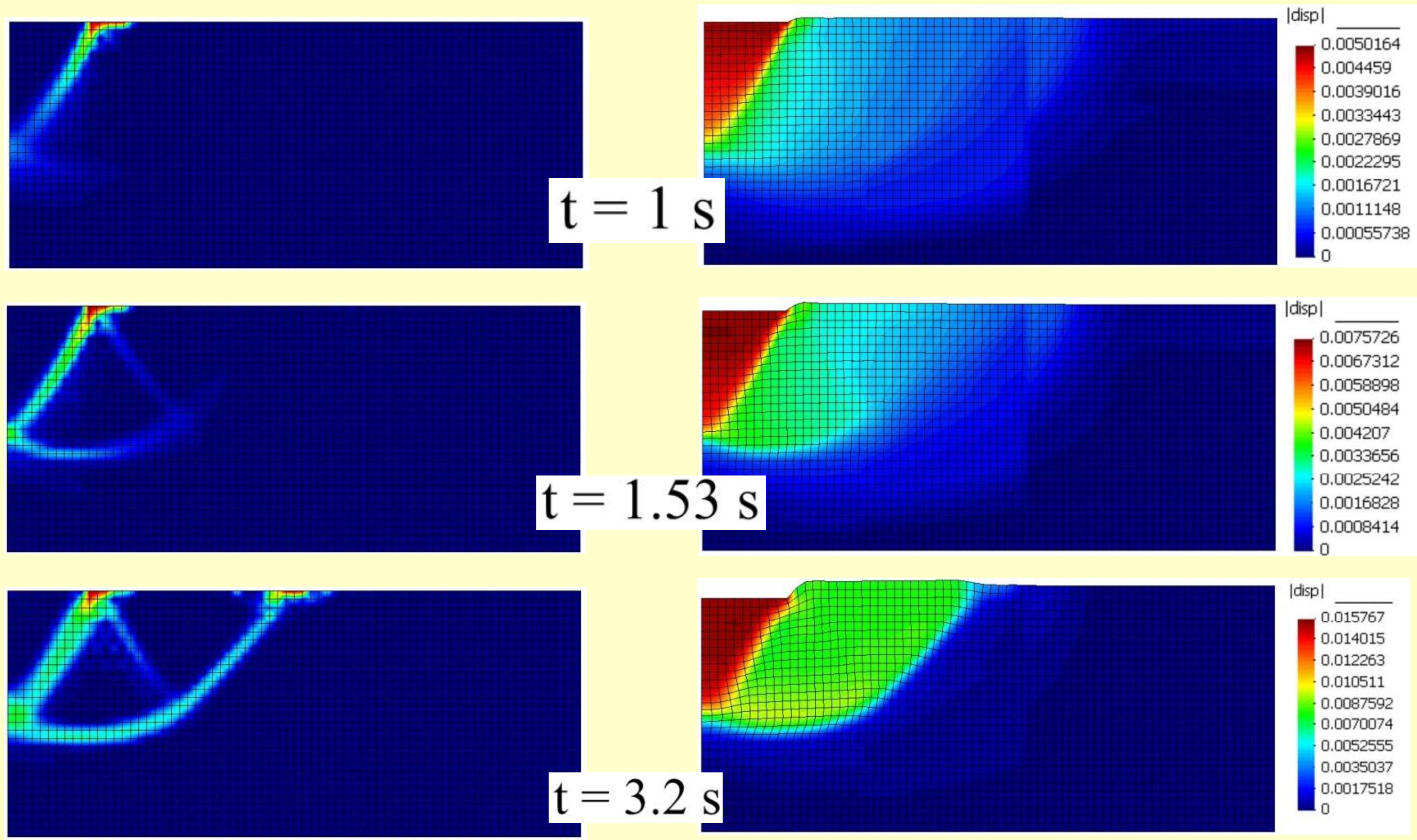
Example 3 – Failure of a shallow stratum under wide strip footing

- Analytical solution: Limit load $$P_{analitycal} = \frac{(2+\pi)}{\sqrt{3}} \cdot b \cdot Y_0$$ ➔ $P_{analitycal}$ = 89027 N



Example 3 – Failure of a shallow stratum under wide strip footing

- Viscoplastic deformations and displacements



Conclusions of part 2

- Modification of the initial model to study problems in the framework of the large deformation theory
- Elimination of the Hourglass deformations and new free-surface detection algorithm
- Stabilized model (Shockwave propagation example in one-dimensional bar)
- Allows reproducing localized failure in geomaterial (well-defined shear bands)
- Large deformation (Deformations until 150 %)
- Complicated failure mechanisms
- Accurate model(comparison with analytical solution)

Contents

Fractional step

- 1st order equations: coupled problems

$$\rho\frac{\partial v}{\partial t} = \operatorname{div}\sigma + \rho b$$

$$\frac{\partial \sigma}{\partial t} = D^e \operatorname{grad}^s v - D^e \dot{\varepsilon}^{vp}$$

$$\rho\frac{\partial v}{\partial t} = \operatorname{div}\sigma' - \underline{\operatorname{grad} p_w} + \rho b$$

$$\frac{\partial \sigma}{\partial t} = D^e \operatorname{grad}^s v - D^e \dot{\varepsilon}^{vp}$$

$$\frac{1}{Q^*}\frac{dp_w}{dt} = \operatorname{div}\left(k \operatorname{grad} p_w\right) - \operatorname{div} v$$

Incompressible, impermeable limit

$$\cancel{\frac{1}{Q^*}\frac{dp_w}{dt}} = \cancel{\operatorname{div}\left(k \operatorname{grad} p_w\right)} - \operatorname{div} v$$

$$\operatorname{div}\ v = 0$$

Stabilization with fractional step

$$\rho\frac{\partial v}{\partial t} = -\text{grad}\, p + b$$
$$\text{div}\, v = 0$$

$$\Rightarrow$$

$$\rho\frac{v^* - v^n}{\Delta t} = b$$
$$\rho\frac{v^{n+1} - v^*}{\Delta t} = -\text{grad}\, p^{n+1}$$
$$\text{div}\, v^{n+1} = 0$$

$$\text{div}\left(\frac{v^{n+1} - v^*}{\Delta t}\right) = -\frac{1}{\rho}\text{div}\left(\text{grad}\, p^{n+1}\right) \Rightarrow \text{div}\, v^* = -\frac{\Delta t}{\rho}\nabla^2 p^{n+1}$$

FS1	$\rho\frac{v^* - v^n}{\Delta t} = b$
FS2	$\text{div}\, v^* = -\frac{\Delta t}{\rho}\nabla^2 p^{n+1}$
FS3	$\rho\frac{v^{n+1} - v^*}{\Delta t} = -\text{grad}\, p^{n+1}$

Fractional Step method

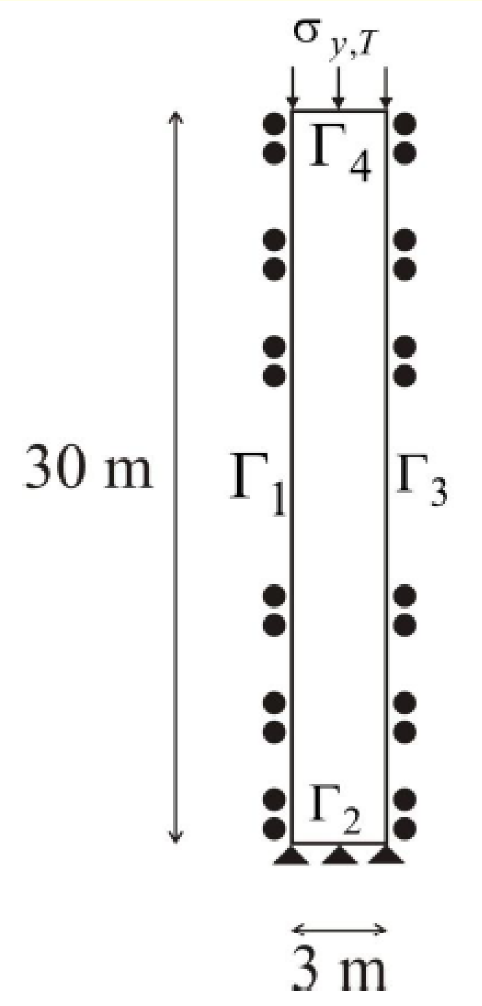
- Initially introduced for incompressible fluid dynamics problems (*Chorin 1968*)

- Technique used to stabilize formulations which use the same order of interpolation for the pore pressure and the velocity. (*Schneider 56 ; Kawahara and Ohmiya 57*)

- Various application in solid and soil dynamics problems with Finite elements (*Zienkiewicz and Wu [60], Zienkiewicz et al [58], Pastor et al [61, 62], Mira et al [5], X.Li et al [63], Mabssout et al [12], White and Borja [6]*)

Example 1 – Saturated soil column under harmonic loading

- Soil column



Example 1 – Saturated soil column under harmonic loading

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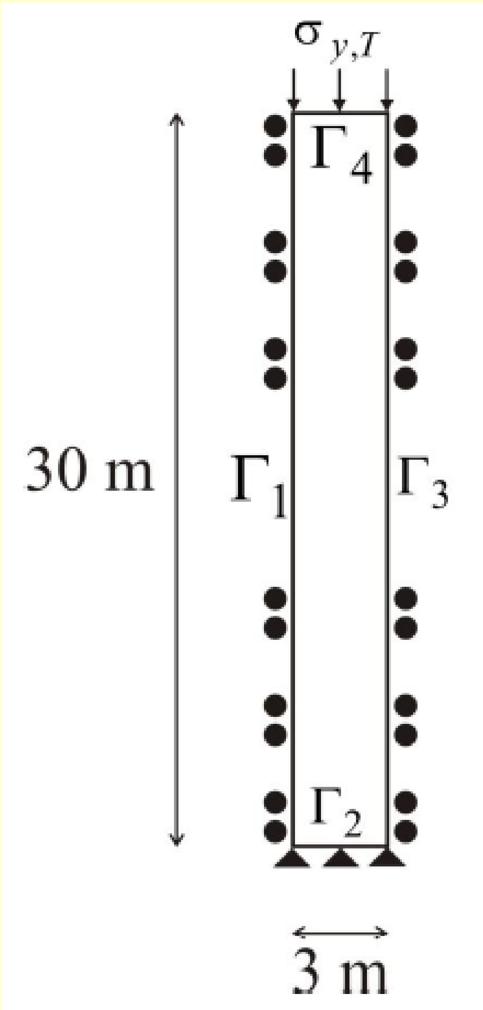


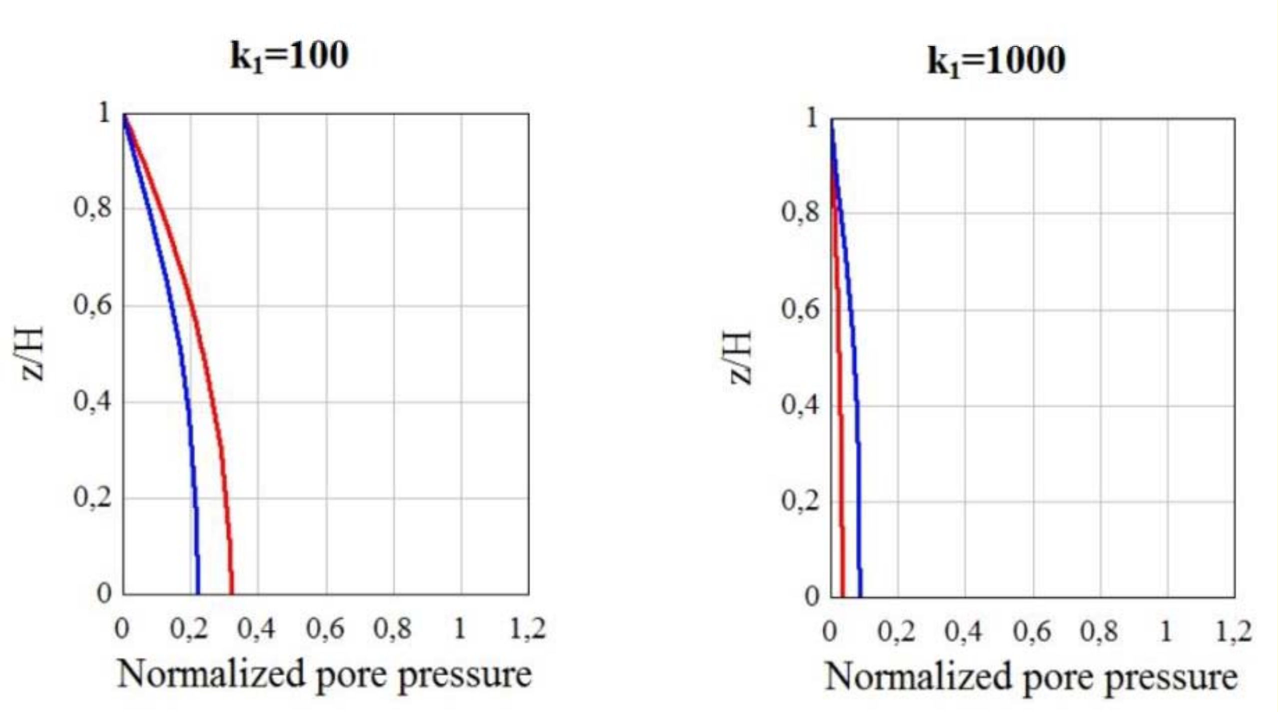
Parameters	Case 1	Case2	Case 3	Case 4	Case 5	Case 6
k_w From:Input data	$5.91 \cdot 10^{-5}$	$5.91 \cdot 10^{-6}$	$5.91 \cdot 10^{-7}$	$5.91 \cdot 10^{-8}$	$5.91 \cdot 10^{-9}$	$5.91 \cdot 10^{-10}$
k' From: $k' = k_w \cdot g \cdot \rho_m$	$9.7 \cdot 10^{-1}$	$9.7 \cdot 10^{-2}$	$9.7 \cdot 10^{-3}$	$9.7 \cdot 10^{-4}$	$9.7 \cdot 10^{-5}$	$9.7 \cdot 10^{-6}$
Q^* From: Input data	$3 \cdot 10^{10}$					
K_w From: Equation (6.59)	$1 \cdot 10^{10}$					
β From: Equation (6.103)	0.6					
V_c^2 From: Equation (6.104)	$1.85 \cdot 10^{7}$					
k_2 From: Equation (6.102)	$5.56 \cdot 10^{-4}$					
k_1 From: Equation (6.101)	10^{3}	10^{2}	10^{1}	1	10^{-1}	10^{-2}

$$k_1 = \frac{k' V_c^2}{g \beta \omega L^2}$$

$$k_2 = \frac{\omega^2 L^2}{V_c^2}$$

Example 1 – Saturated soil column under harmonic loading

- Comparison of he results with analytical solution (*Zienkiewicz et al. 1980*)



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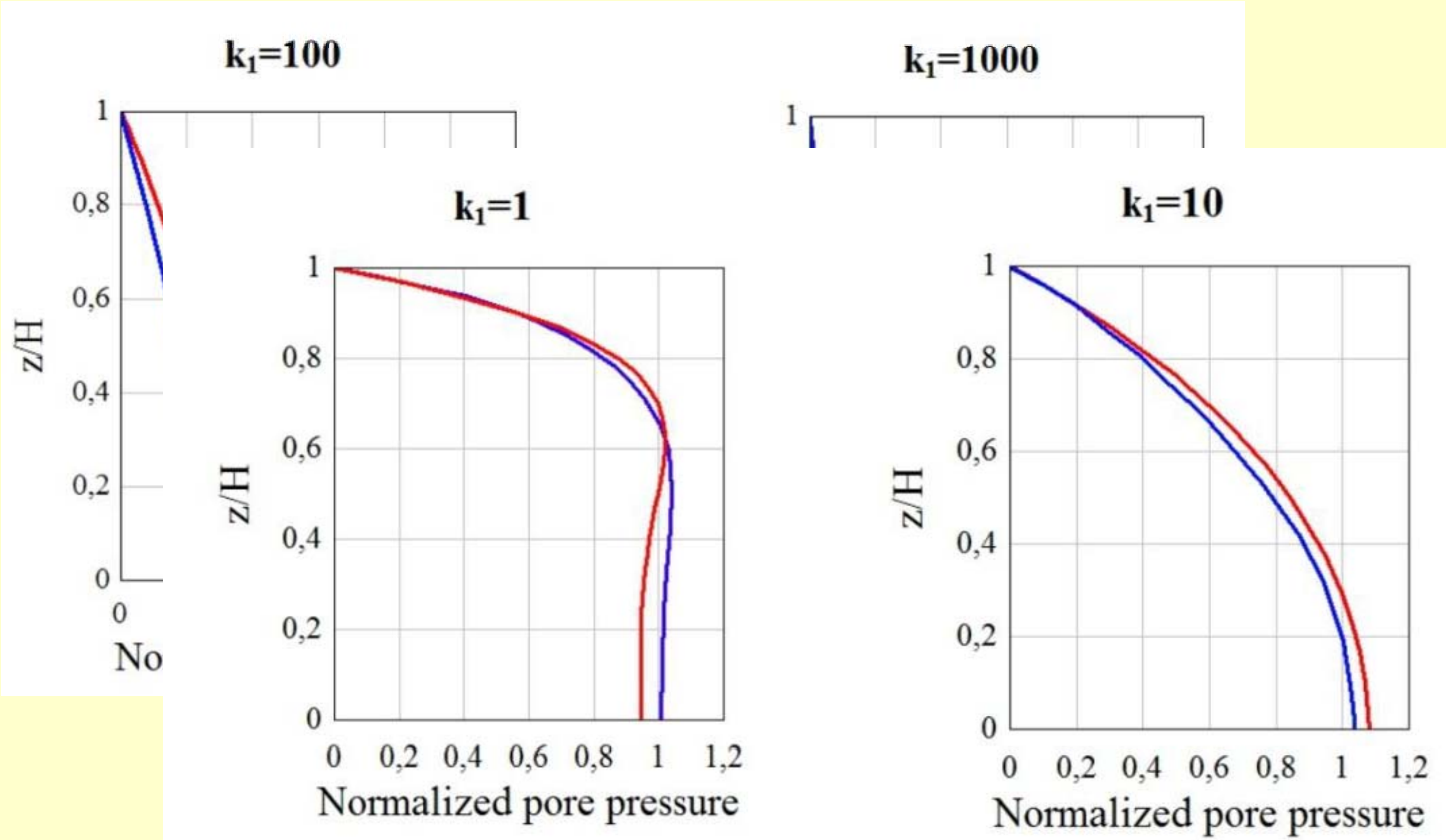

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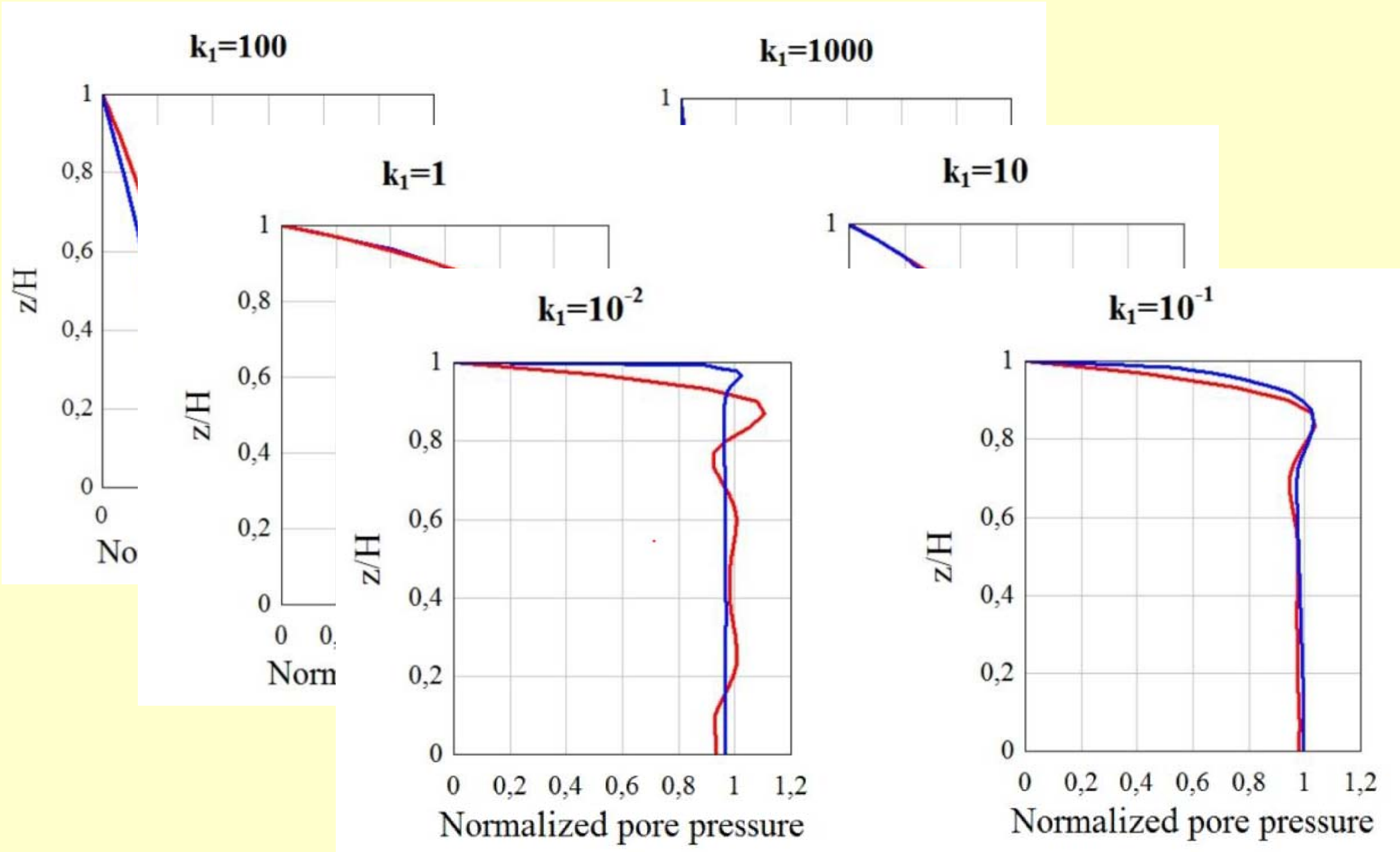
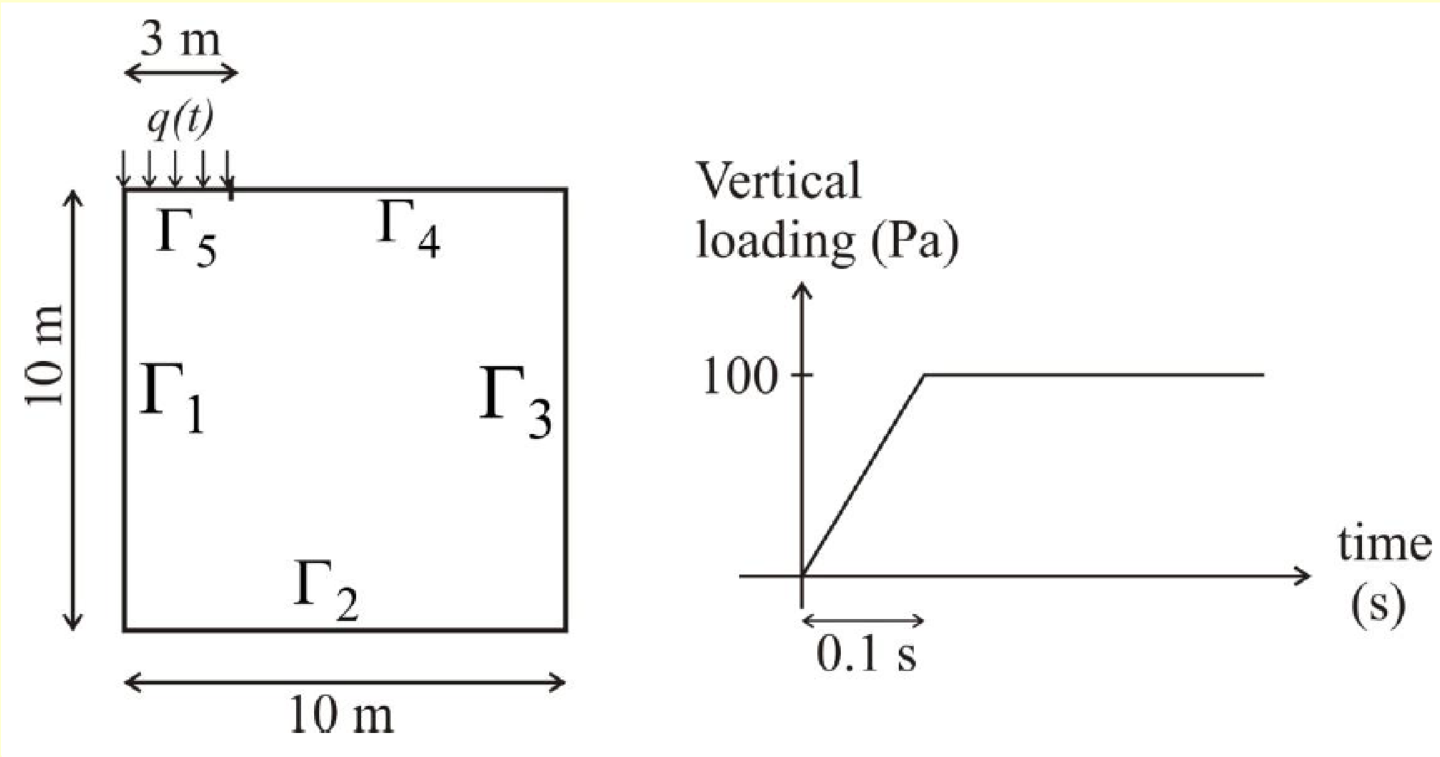


Example 2 – Strip foundation on elastic soil stratum

- Boundary condition



Example 2 – Strip foundation on elastic soil stratum

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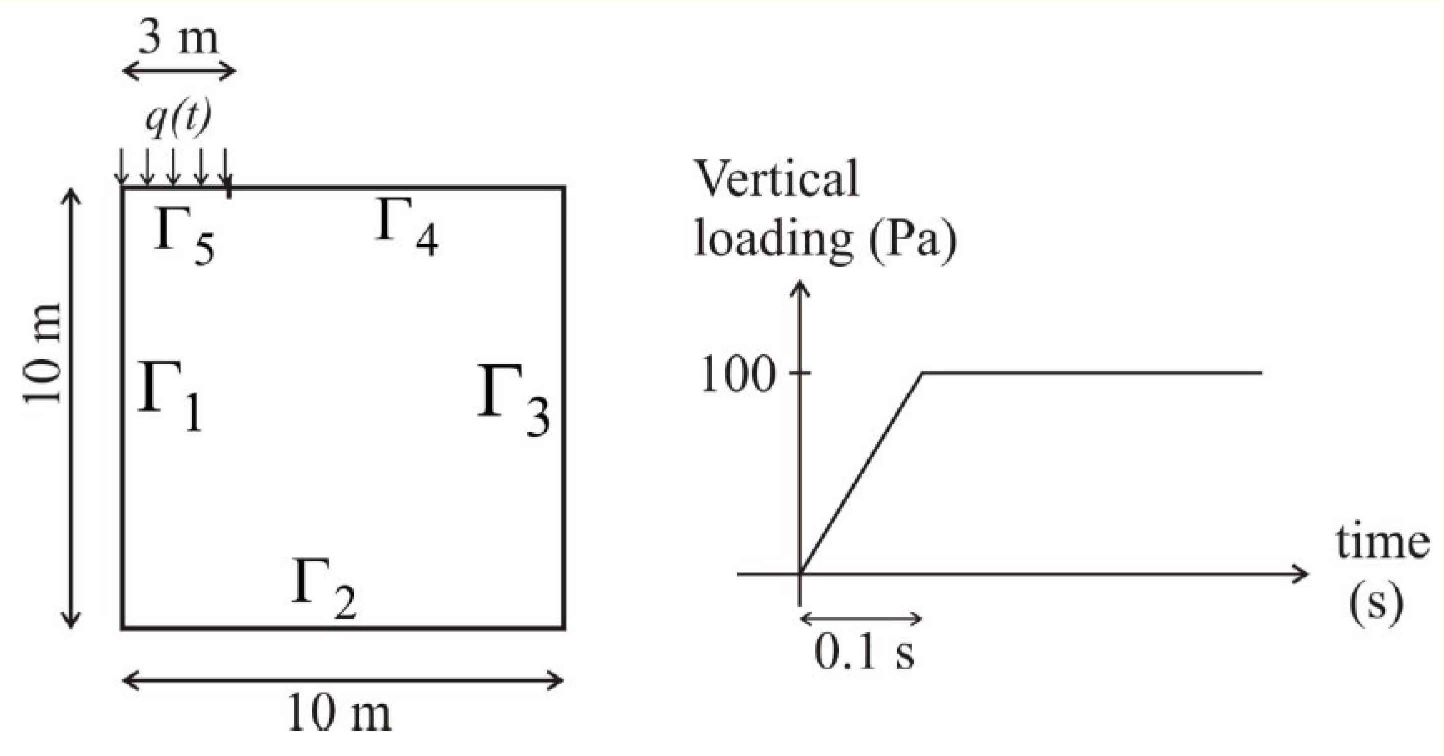
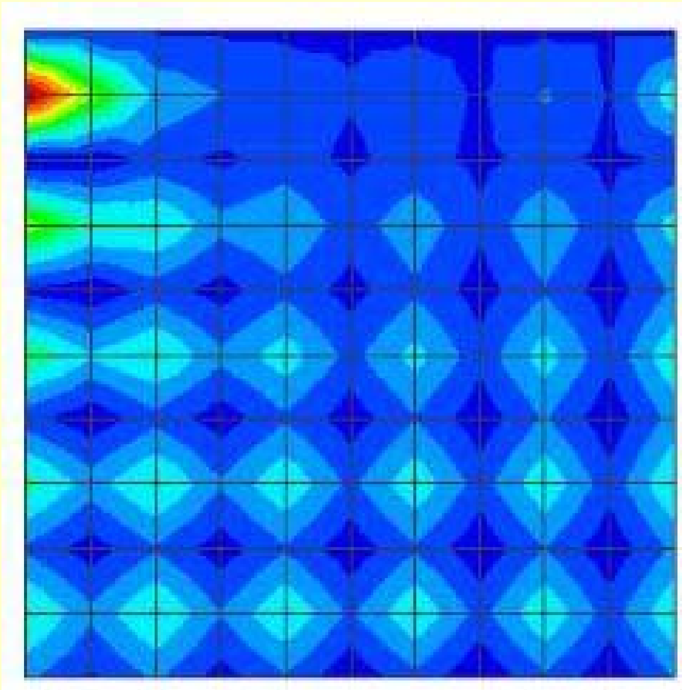


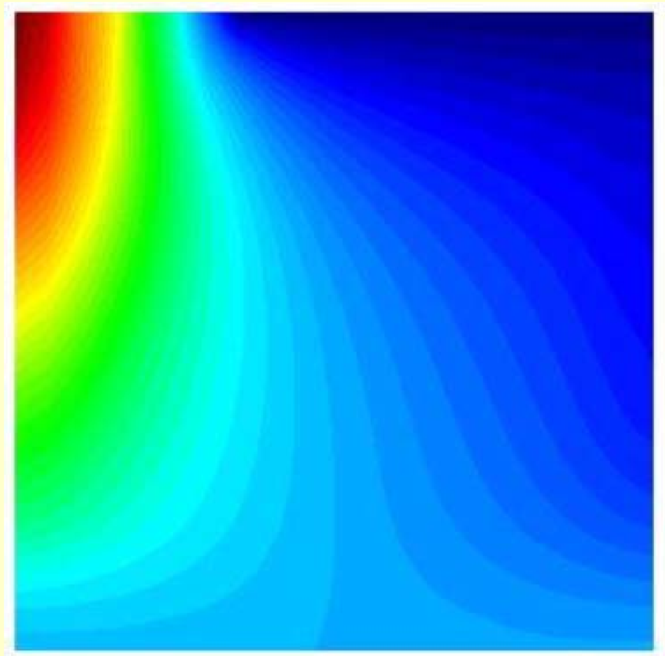
- Elastic material
- Incompressibility condition: $k_w = 1 \cdot 10^{-7}\ \mathrm{ms^{-1}}$

Example 2 – Strip foundation on elastic soil stratum

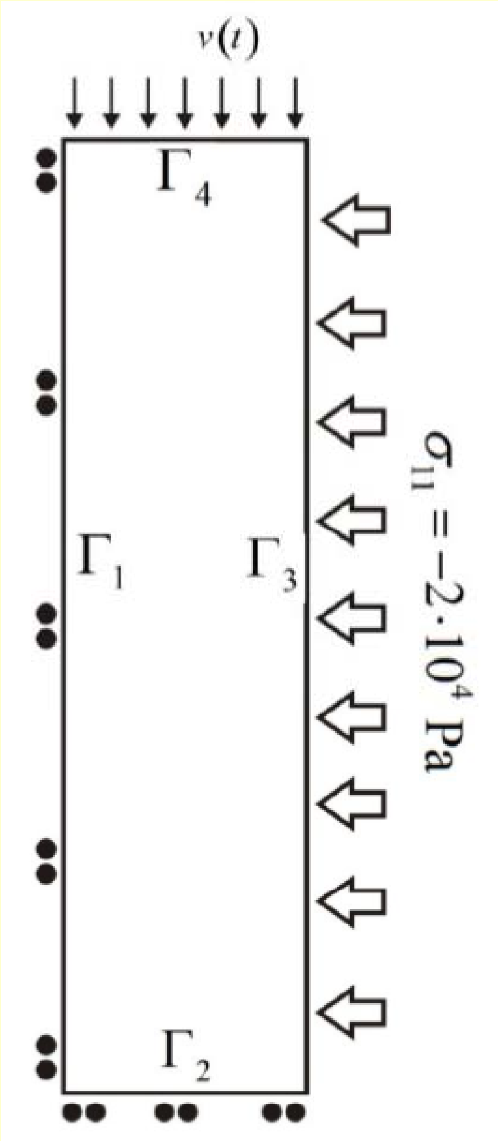
- Pore pressure obtained with a classical finite element formulation

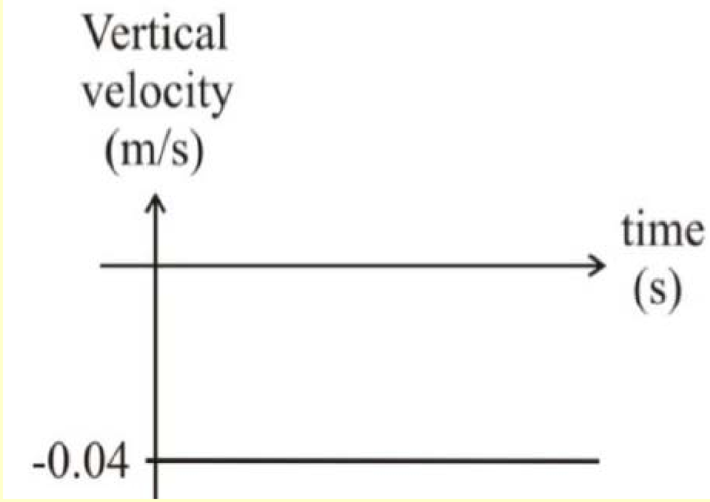
- Pore pressure obtained with the *v*-p_w Runge-Kutta Taylor-SPH model





Example 3 – Strain localization in saturated soil

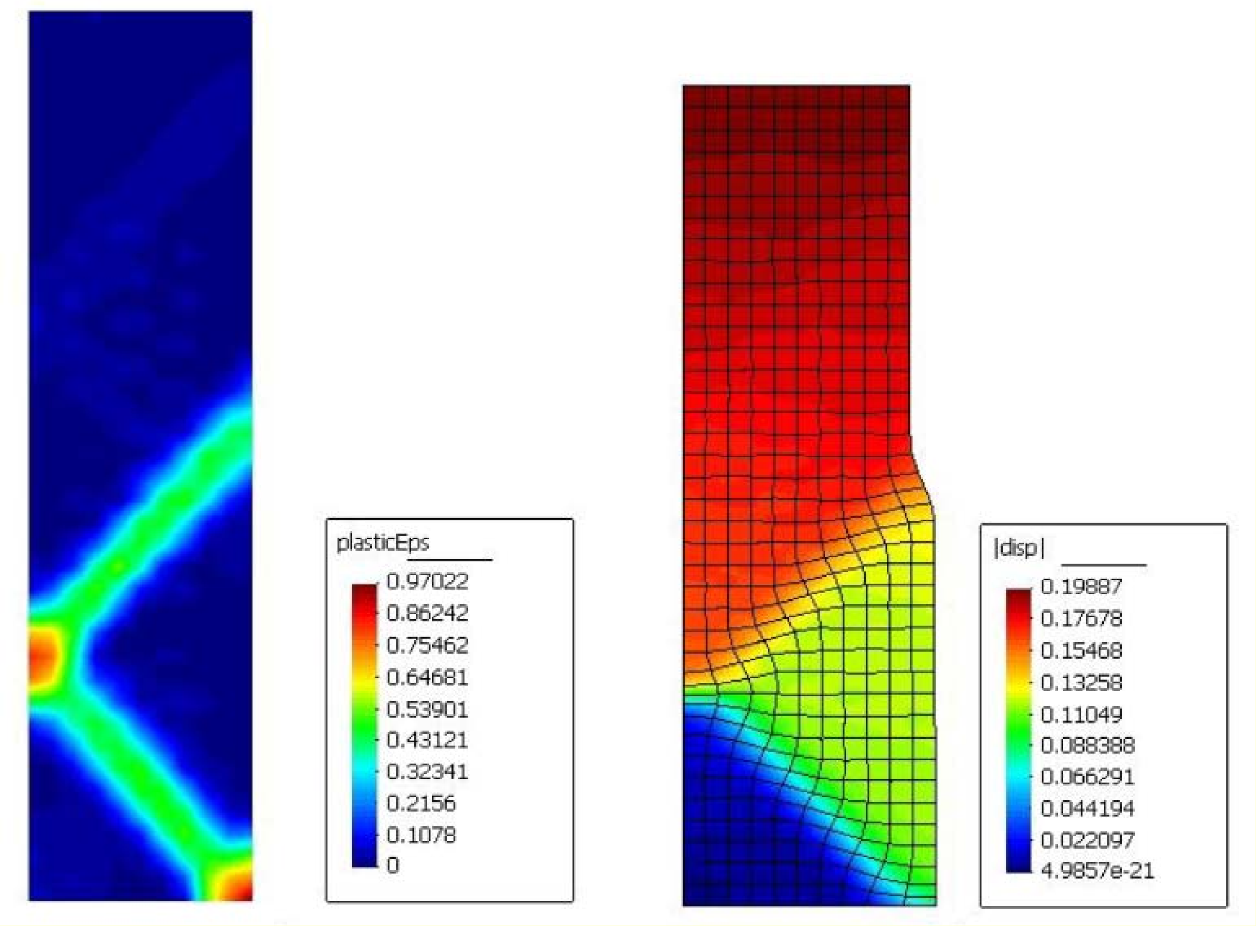




- Viscoplastic material (Cam Clay)
- Permeability $k_w = 1 \cdot 10^{-6}\ \mathrm{ms^{-1}}$

Example 3 – Strain localization in saturated soil

- Viscoplastic deformations and displacements



- **Localization in shear bands**

Conclusions

- Modification of the initial model to study coupled problems in the framework of the small deformation theory

- Based on the Biot-Zienkiewicz theory and the notion of pore pressure and effective stress

- Accurate model (comparison with analytical solution)

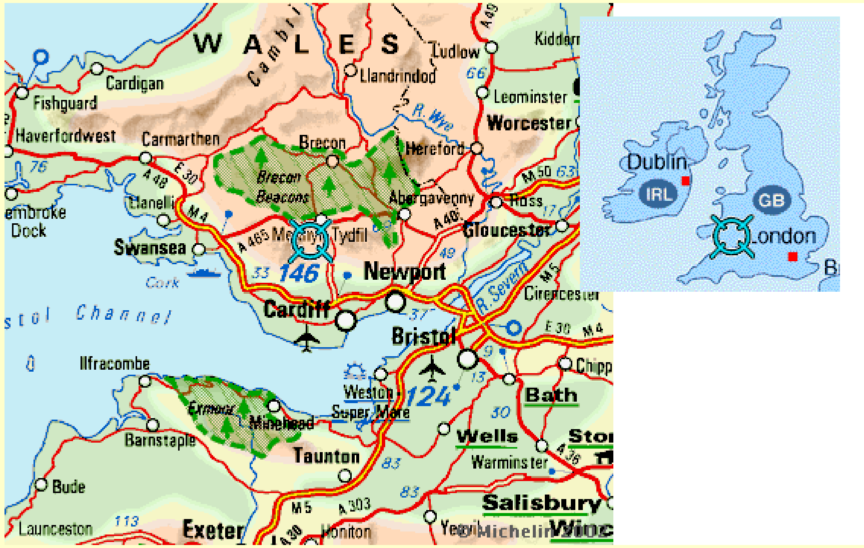
- Stabilized model

- Reproduce localized failure in saturated geomaterials with well-defined shear bands

Contents

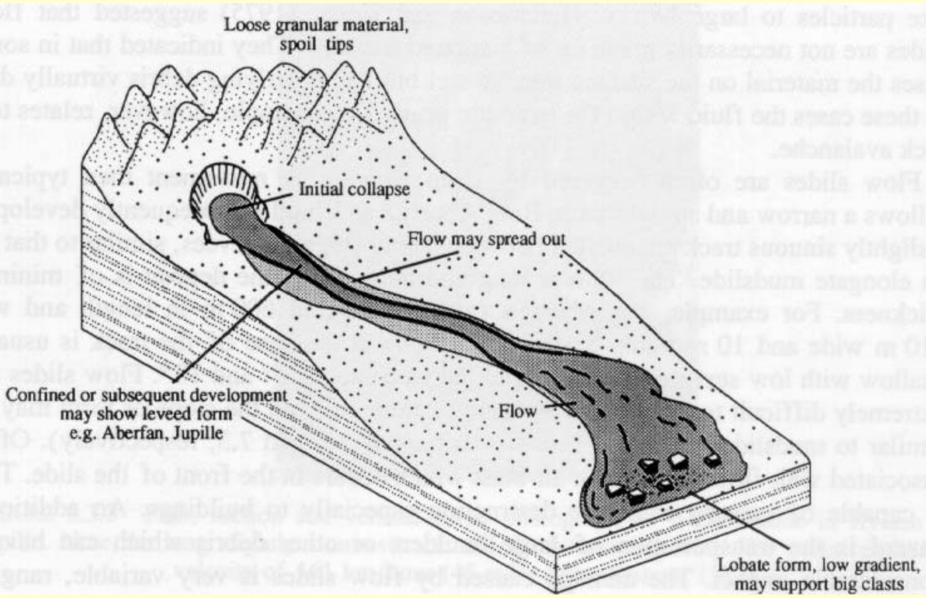
- Introduction
- SPH (I) Basic Model
- SPH (II) Large deformation
- SPH (III) Coupled v-p_w
- SPH (IV) Depth integrated models h-v-p_w

Benchmarks: Flowslide at Aberfan (21 Oct 1966)



Tip of a loose colliery
waste
200 m above of Aberfan
slope 25°
100000 m^3
(144 dead)

Flow slides



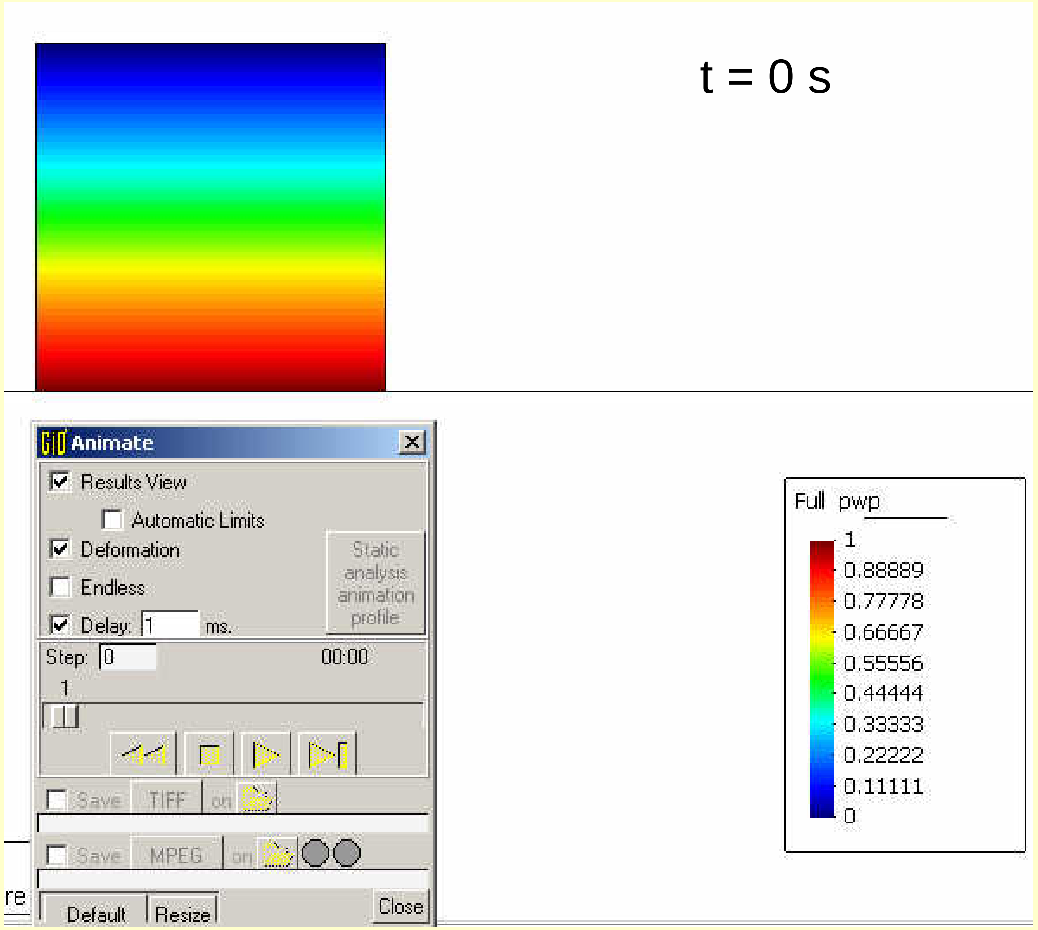
(Aberfan, 1966)

Dam break problem: Saturated soil

$$\tan\phi = 1.0 \quad C_v = 2.10^{-4} m^2 s^{-1}$$

10 m

t = 0 s



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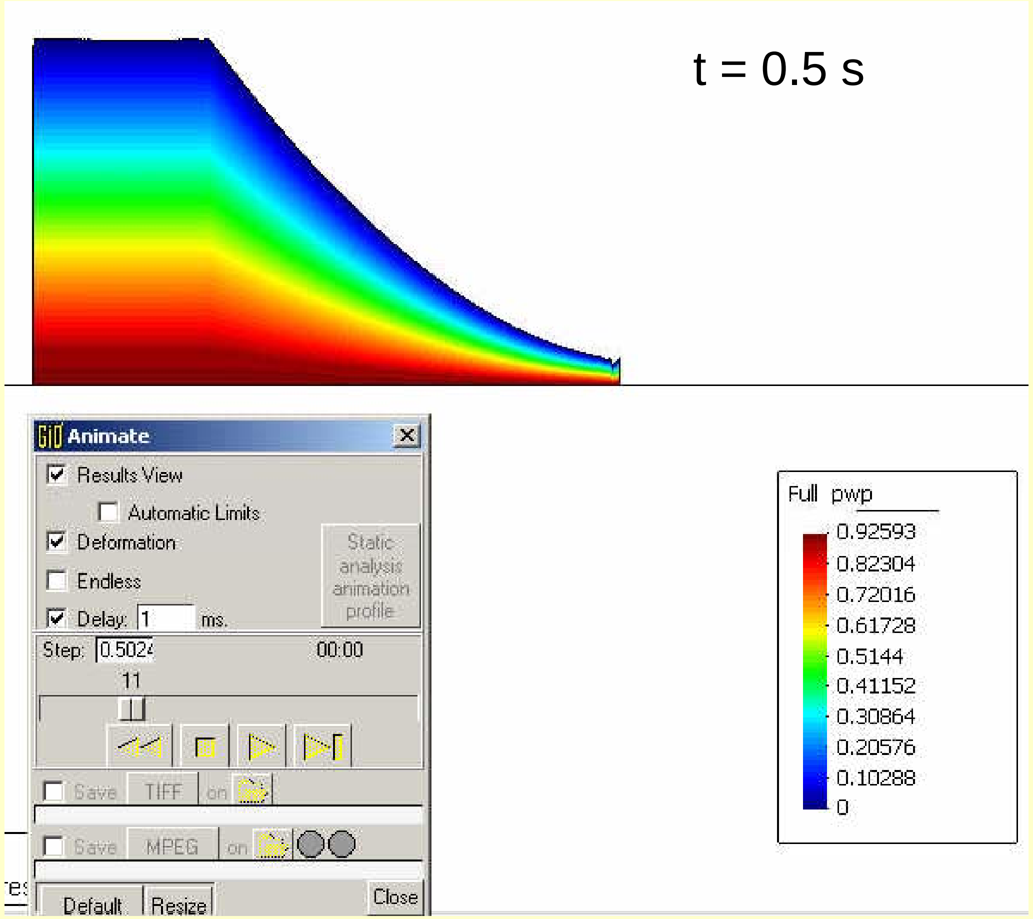
$$\tan\phi = 1.0 \quad C_v = 2.10^{-4}\, m^2 s^{-1}$$



Dam break problem: Saturated soil

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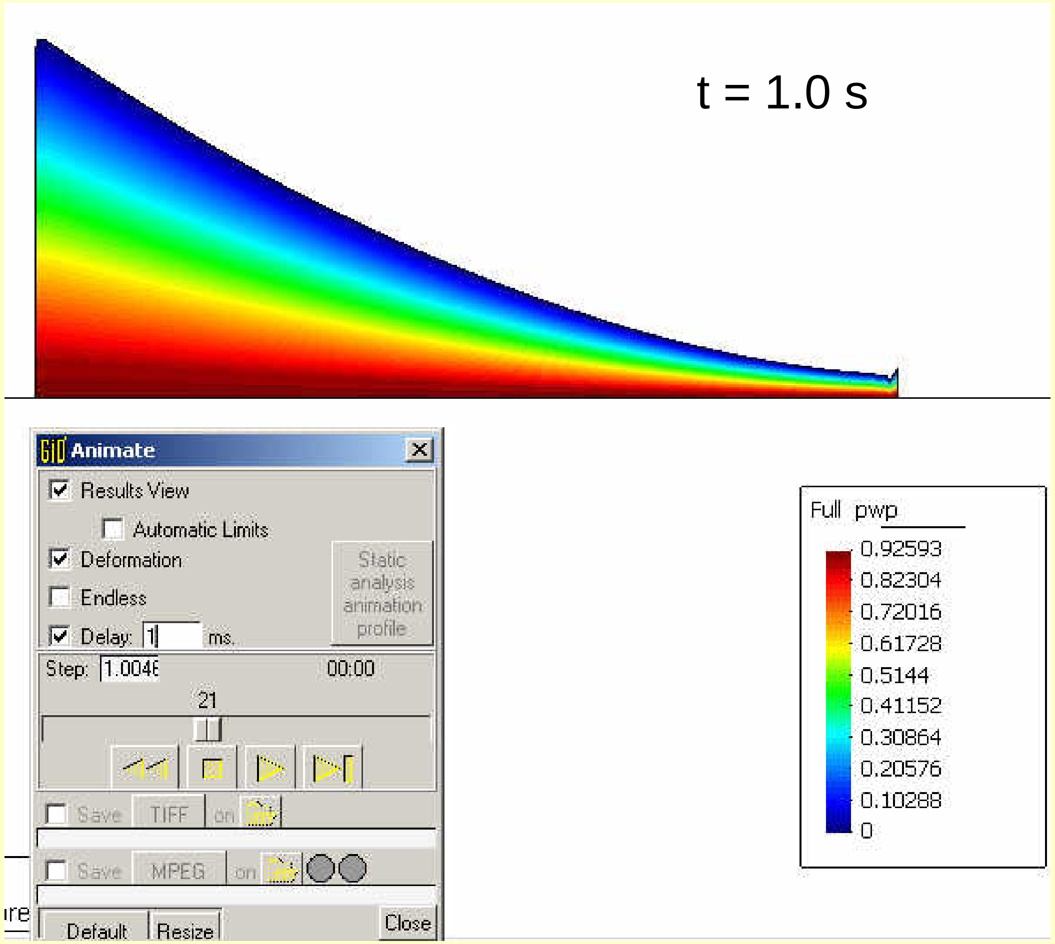
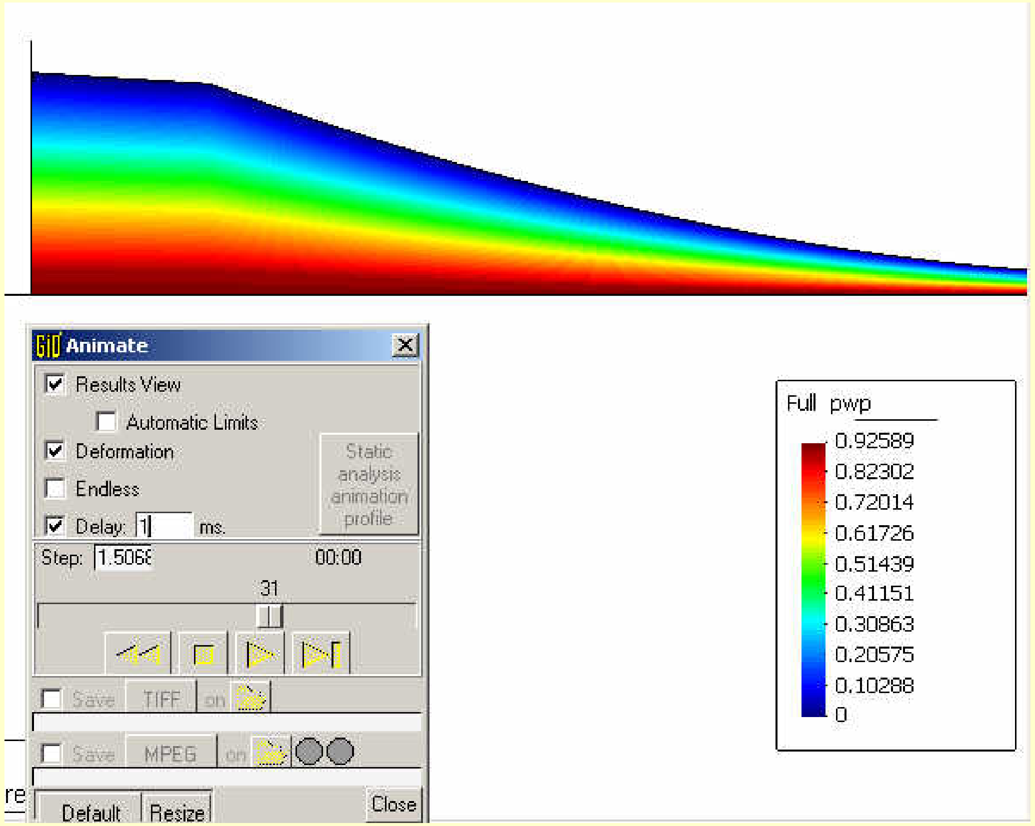


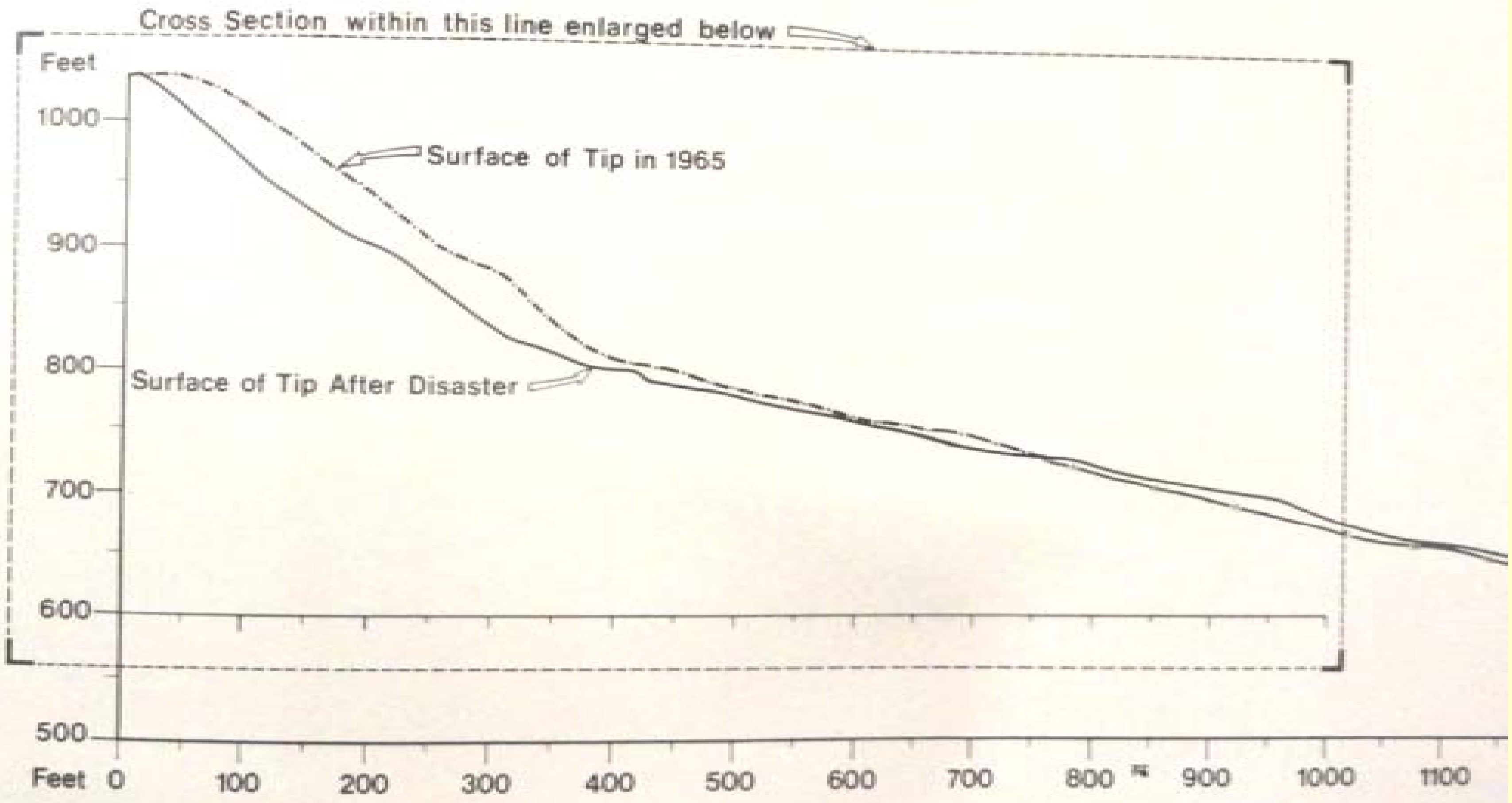
Dam break problem: Saturated soil

$$\tan\phi = 1.0 \quad C_v = 2.10^{-4}\, m^2 s^{-1}$$

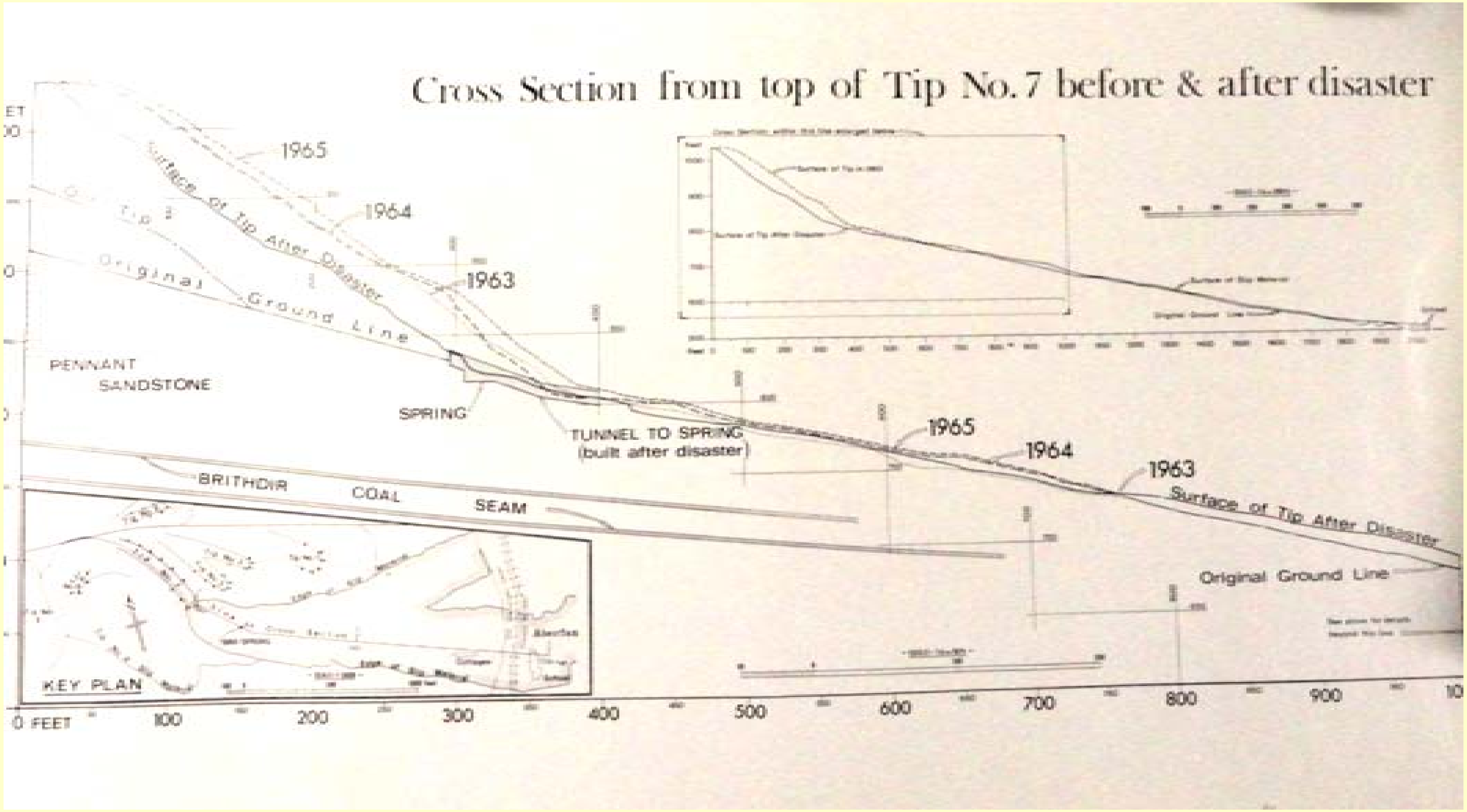
t = 1.5 s

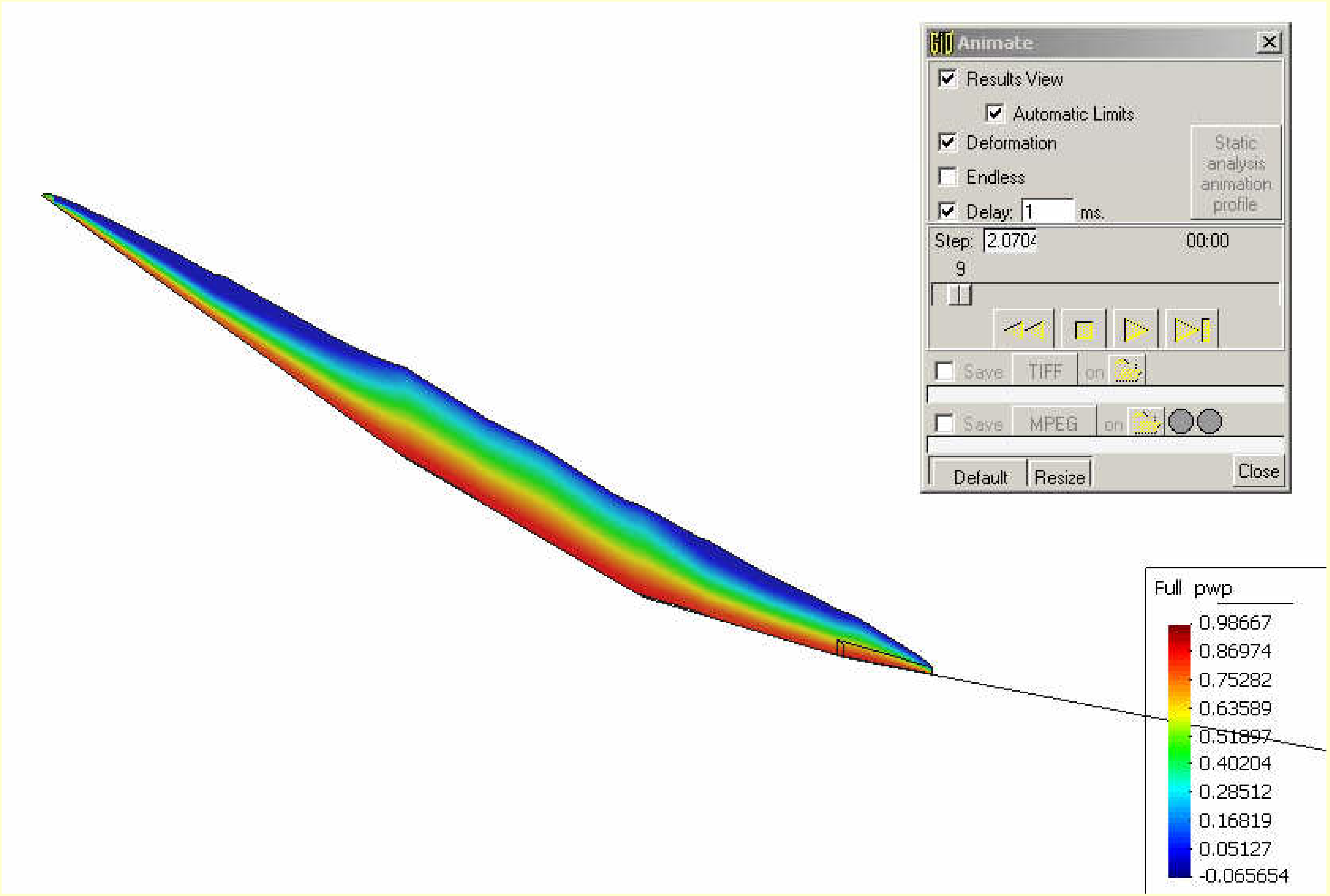


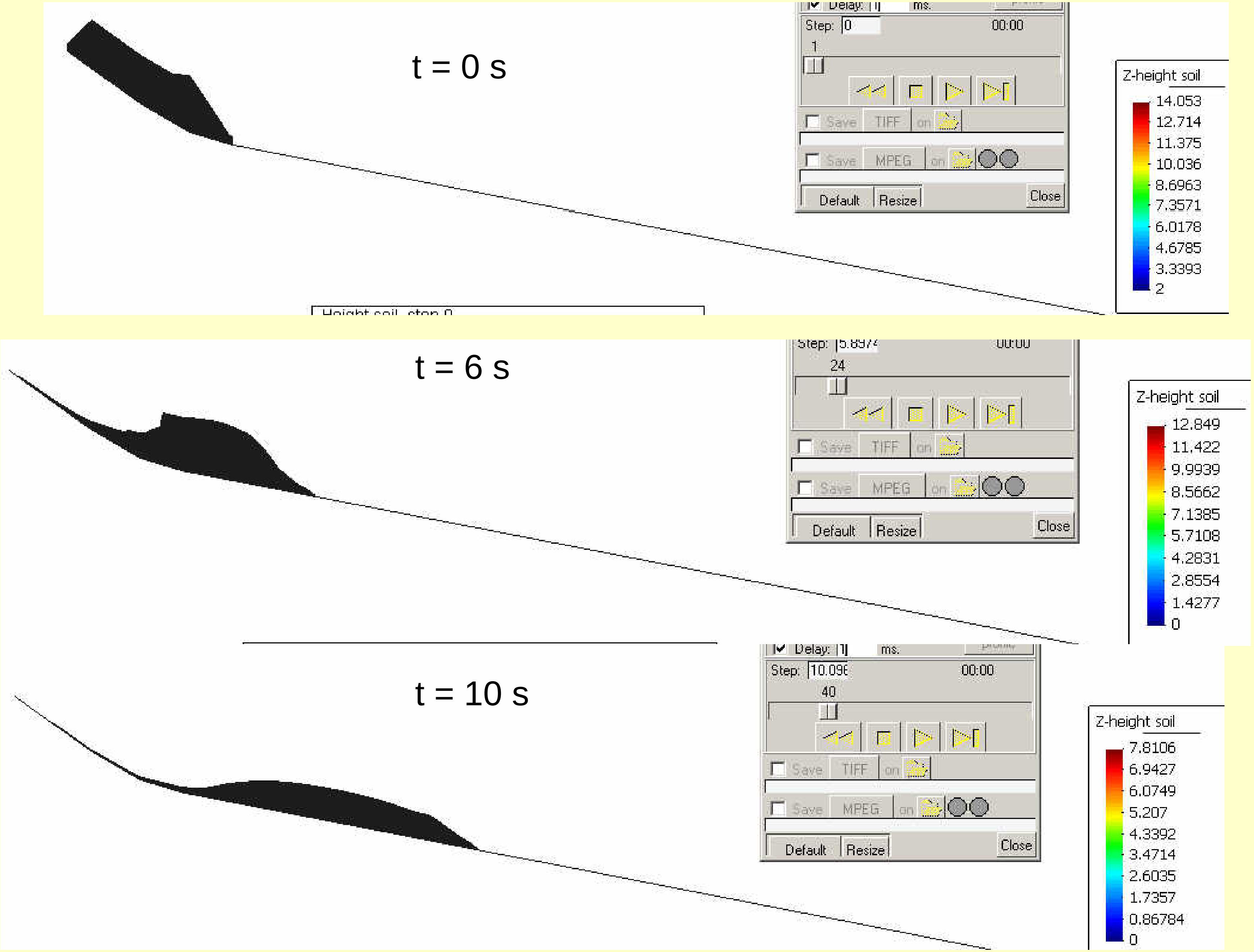
Cross sections of Aberfan Tip 7 before and after the flowslide



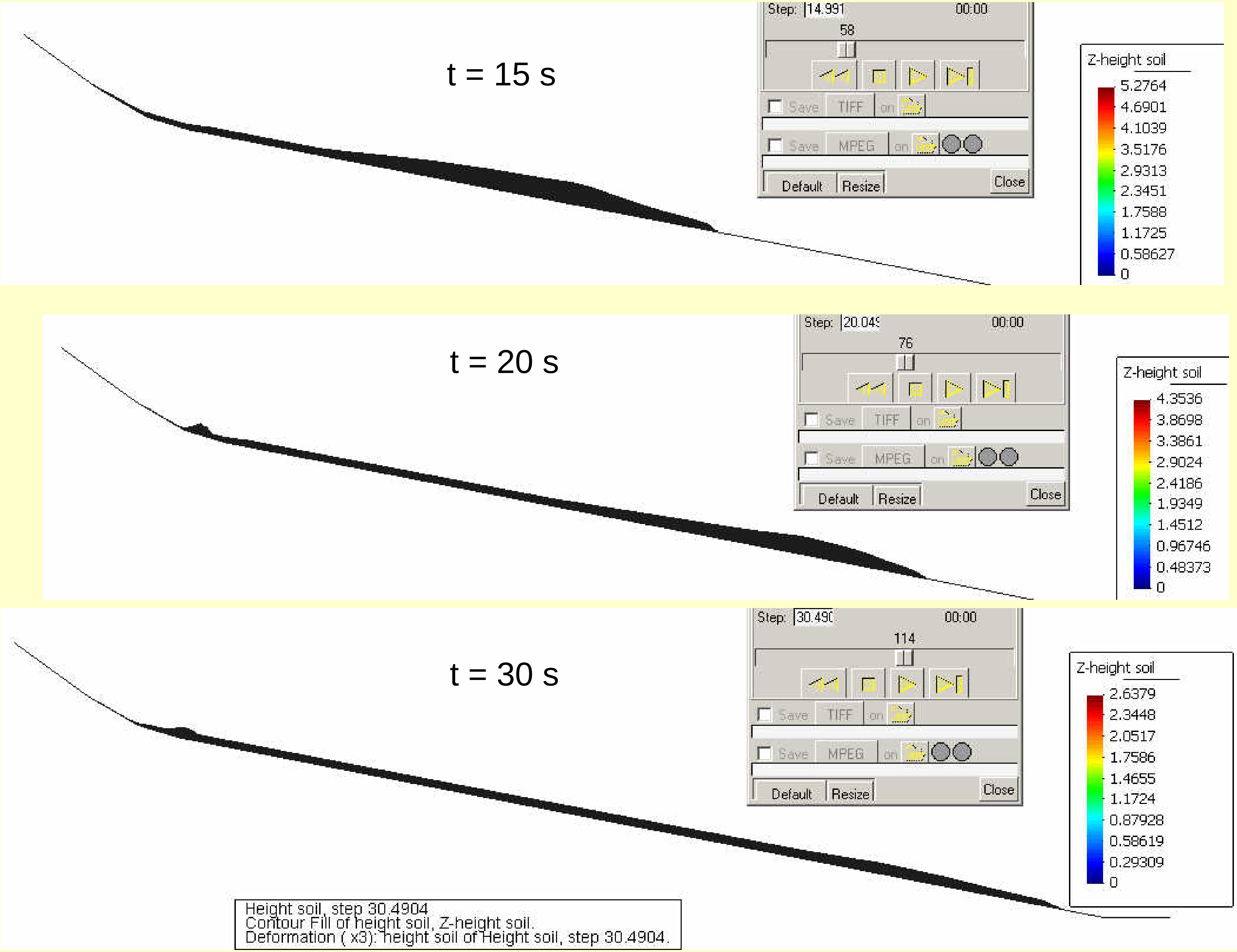
Cross sections of Aberfan Tip 7 before and after the flowslide

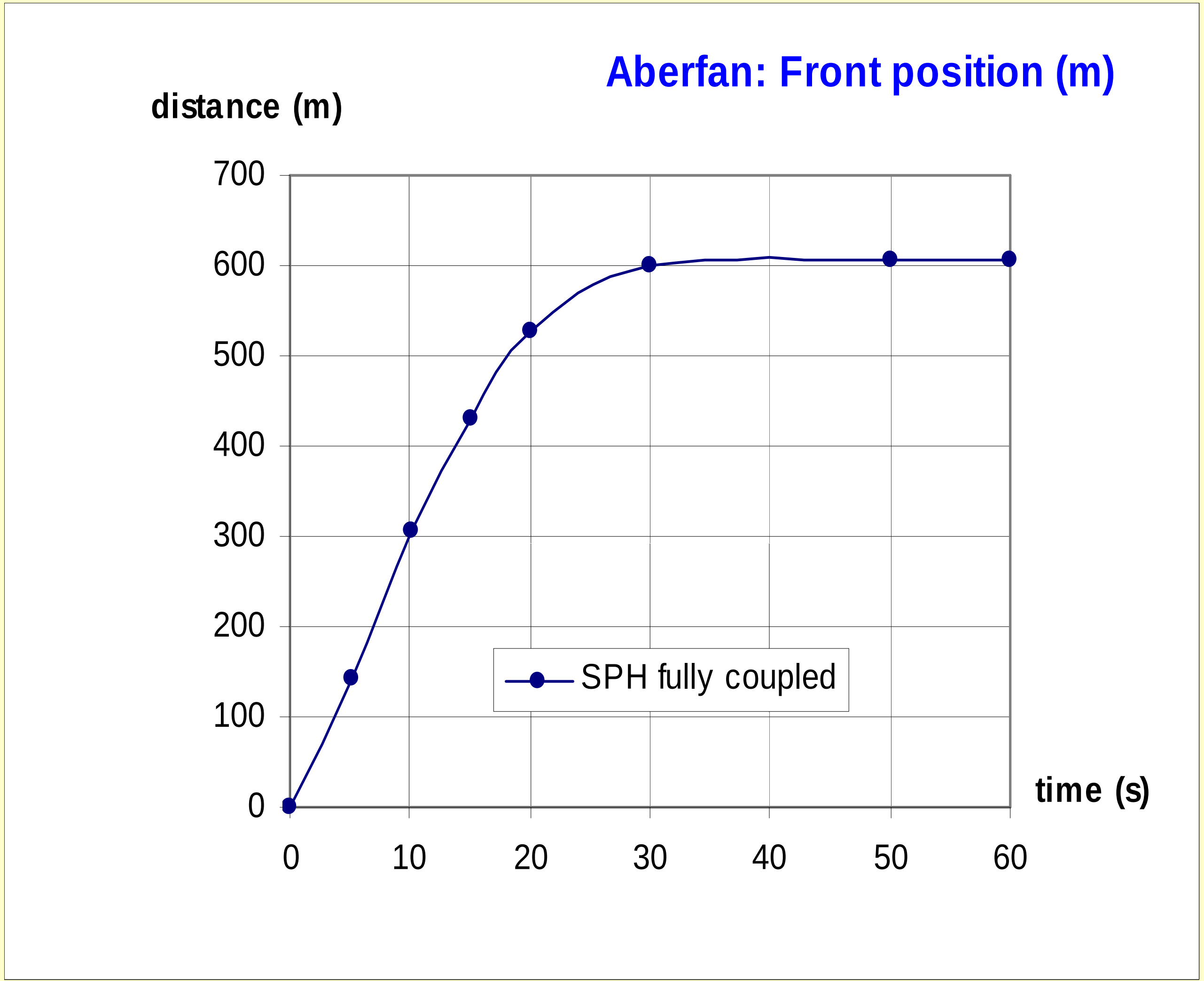



Animate
Results View
Automatic Limits
Deformation
Endless
Delay: 1 ms.
Static analysis animation profile
Step: 2.0704
00:00
9
Save TIFF on
Save MPEG on
Default
Resize
Close
Full pwp
0.98667
0.86974
0.75282
0.63589
0.51897
0.40204
0.28512
0.16819
0.05127
-0.065654

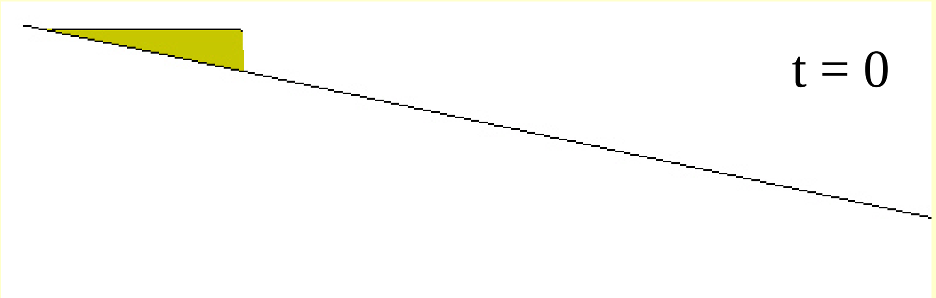


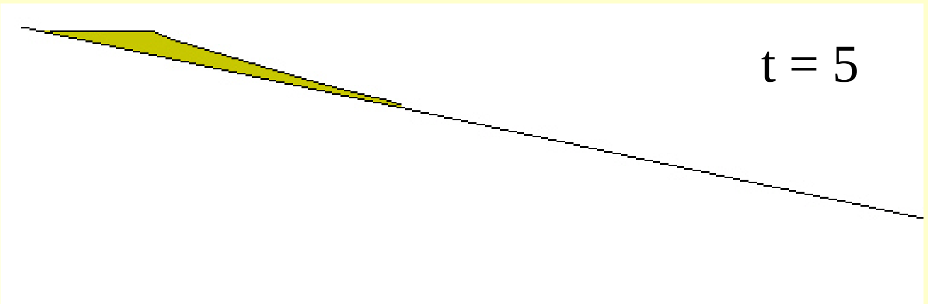
t = 0 s
Delay: 1 ms.
Step: 0
00:00
1
Save TIFF on
Save MPEG on
Default
Resize
Close
Z-height soil
14.053
12.714
11.375
10.036
8.6963
7.3571
6.0178
4.6785
3.3393
2
t = 6 s
24
Save TIFF on
Save MPEG on
Default
Resize
Close
Z-height soil
12.849
11.422
9.9939
8.5662
7.1385
5.7108
4.2831
2.8554
1.4277
0
t = 10 s
Delay: 1 ms.
Step: 10.096
00:00
40
Save TIFF on
Save MPEG on
Default
Resize
Close
Z-height soil
7.8106
6.9427
6.0749
5.207
4.3392
3.4714
2.6035
1.7357
0.86784
0

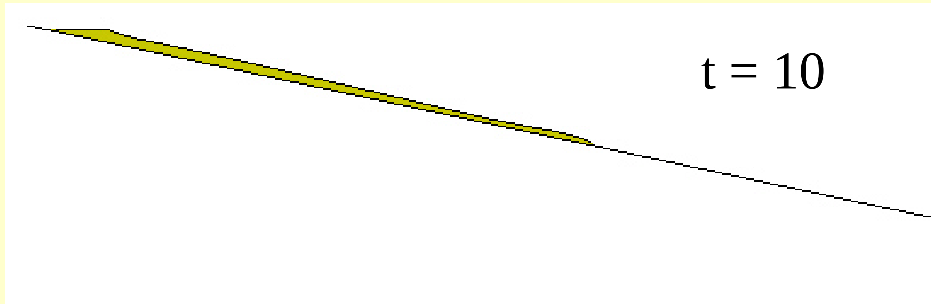

t = 15 s
Step: 14.991
00:00
58
Save TIFF on
Save MPEG on
Default Resize Close
Z-height soil
5.2764
4.6901
4.1039
3.5176
2.9313
2.3451
1.7588
1.1725
0.58627
0
t = 20 s
Step: 20.049
00:00
76
Save TIFF on
Save MPEG on
Default Resize Close
Z-height soil
4.3536
3.8698
3.3861
2.9024
2.4186
1.9349
1.4512
0.96746
0.48373
0
t = 30 s
Step: 30.490
00:00
114
Save TIFF on
Save MPEG on
Default Resize Close
Z-height soil
2.6379
2.3448
2.0517
1.7586
1.4655
1.1724
0.87928
0.58619
0.29309
0
Height soil, step 30.4904
Contour Fill of height soil, Z-height soil.
Deformation (x3): height soil of Height soil, step 30.4904.

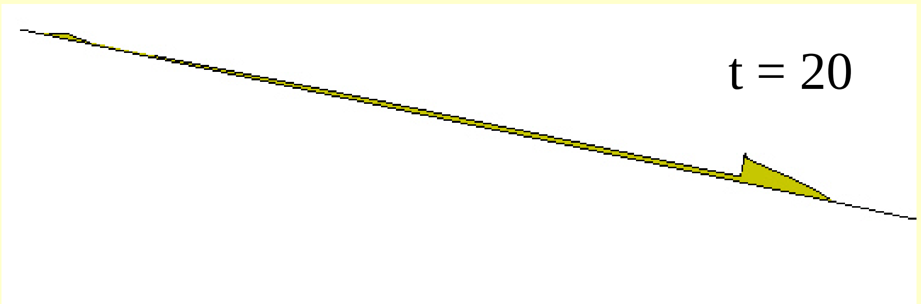

Aberfan: Front position (m)
distance (m)
700
600
500
400
300
200
100
0
0
10
20
30
40
50
60
time (s)
SPH fully coupled

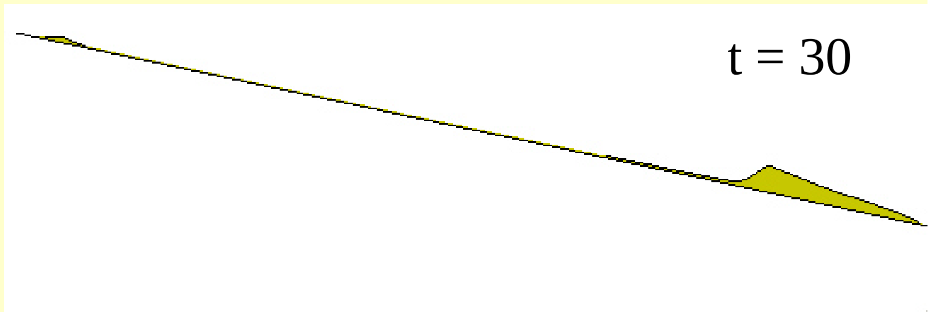
Aberfan Flowslide   (1966)











Thanks for your attention