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Modelling of instabilities and bifurcation in Geomechanics, 6-8 Oct. 2016

Material instability and bifurcation analysis

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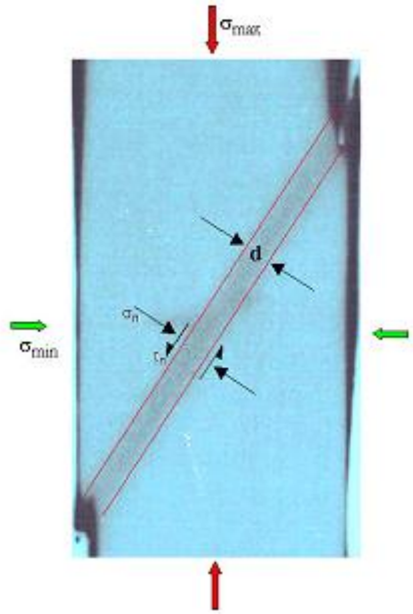
Strain localization analysis

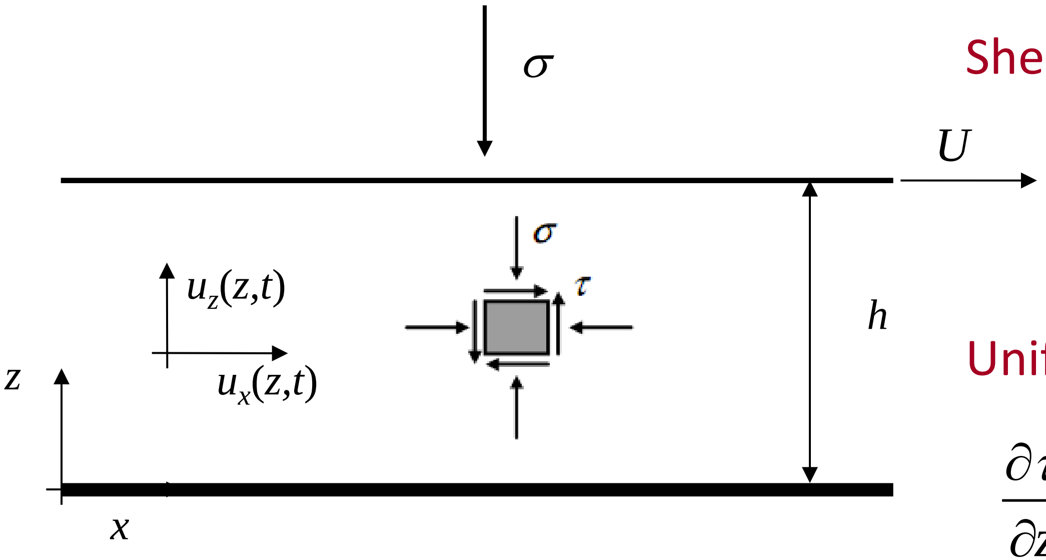


The strain localization analysis consists in searching the incipient of a shear band in a solid as a *mathematical bifurcation condition* for the deformation field.

The strain localization phenomenon is understood as the appearance of a *discontinuity in strain rates* which marks the onset of non-uniform response.

1D Localization in a layer under simple shear

Palmer & Rice (1973), Rice (1975), Vardoulakis (1976), Bésuelle & Rudnicki (2004)….



Shear strain and volume strain

$$\gamma = \frac{\partial u_x}{\partial z} \qquad \varepsilon = \frac{\partial u_z}{\partial z}$$

Uniform state of stress in the layer

$$\frac{\partial \tau}{\partial z} = 0 \qquad \frac{\partial \sigma}{\partial z} = 0$$

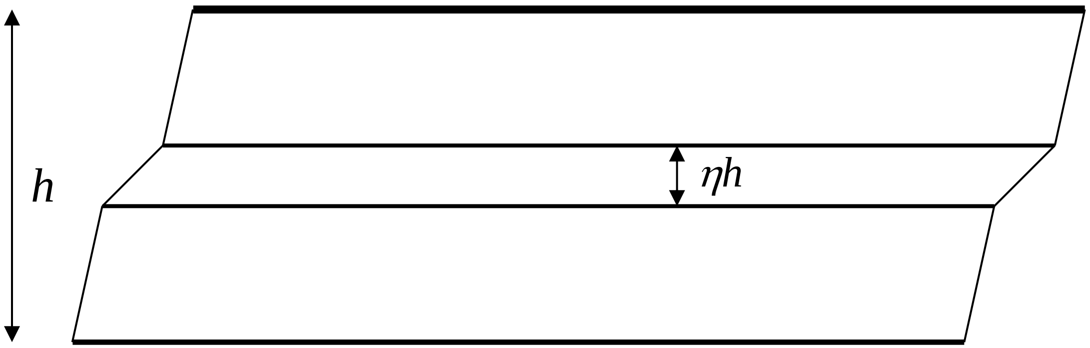
One trivial solution: homogeneous deformation: $\gamma = \dfrac{U}{h}$

We assume that the deformation in the layer is homogeneous up to $\gamma = \gamma_0$

Possibility of non homogeneous deformation in the next increment?

1D Localization in a layer under simple shear

Non uniform deformation in a band of thickness ηh

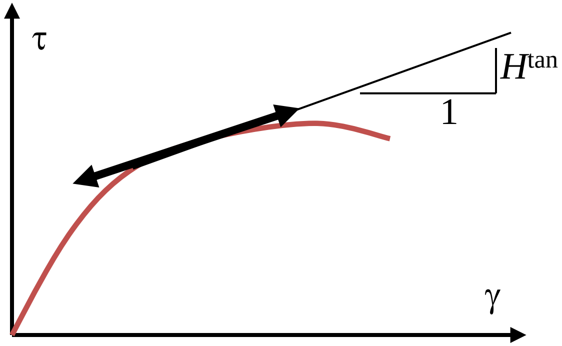


$$\dot{\gamma}^{\text{band}} = \dot{\gamma}^{\text{out}} + \Delta\dot{\gamma}$$

$$\frac{\partial \tau}{\partial z} = 0 \Rightarrow \dot{\tau}^{\text{band}} = \dot{\tau}^{\text{out}}$$

Under constant σ, assuming that the tangent modulus is the same inside and outside the band (loading and unloading) *(continuous bifurcation)*

$$H^{\tan}(\gamma_0)\dot{\gamma}^{\text{band}} = H^{\tan}(\gamma_0)\dot{\gamma}^{\text{out}} \Rightarrow H^{\tan}(\gamma_0)\Delta\dot{\gamma} = 0$$

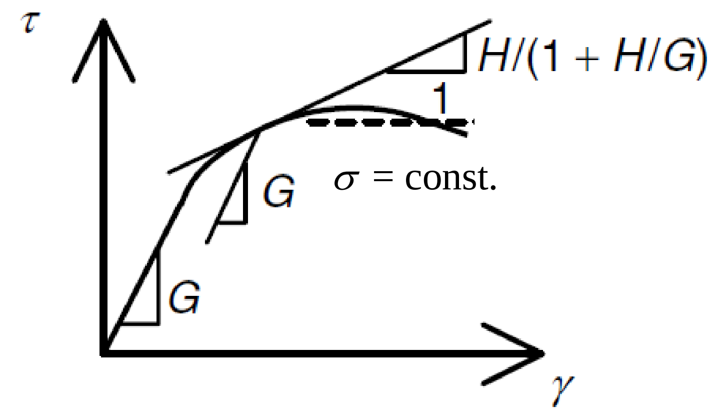


For $H^{\tan}(\gamma_0) > 0, \Delta\dot{\gamma} = 0$

First occurrence of non homogeneous deformation for $H^{\tan}=0$ (peak of the stress strain curve)

1D Localization in a layer under simple shear

Elasto-plastic constitutive equations



$$\dot{\gamma} = \frac{1}{G}\dot{\tau} + \dot{\gamma}^{p} \qquad \dot{\gamma}^{p} = \frac{1}{H}\left(\dot{\tau} - \mu\dot{\sigma}\right)$$

For elasto-plastic solids the response differs in loading and unloading

$$H_{\text{out}}^{\tan} = G \text{ (elastic unloading),}$$

$$H_{\text{band}}^{\tan} = H/\left(1+H/G\right) \text{ (plastic loading)}$$

$$G\dot{\gamma}^{\text{out}} = \frac{H}{1+H/G}\dot{\gamma}^{\text{band}}$$

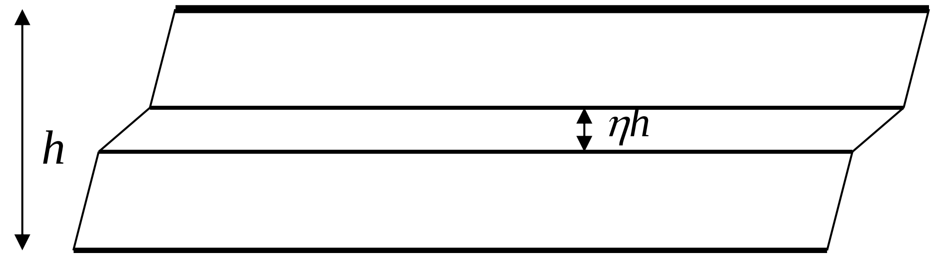
H is initially positive. First occurrence of loss of homogeneous deformation is obtained again for $H=0$

$$\text{For } H=0, \dot{\gamma}^{\text{out}}=0$$

At the onset of localization, the material outside the incipient band behaves as a rigid solid

1D Localization in a layer under simple shear

The non homogeneous solution for $H^{\tan} \leq 0$ is not unique



$$\dot{U} = \eta h \dot{\gamma}^{\text{band}} + (1-\eta) h \dot{\gamma}^{\text{out}}$$

$$\dot{\tau}^{\text{band}} = \dot{\tau}^{\text{out}} \Rightarrow H_{\tan}^{\text{band}} \dot{\gamma}^{\text{band}} = H_{\tan}^{\text{out}} \dot{\gamma}^{\text{out}}$$

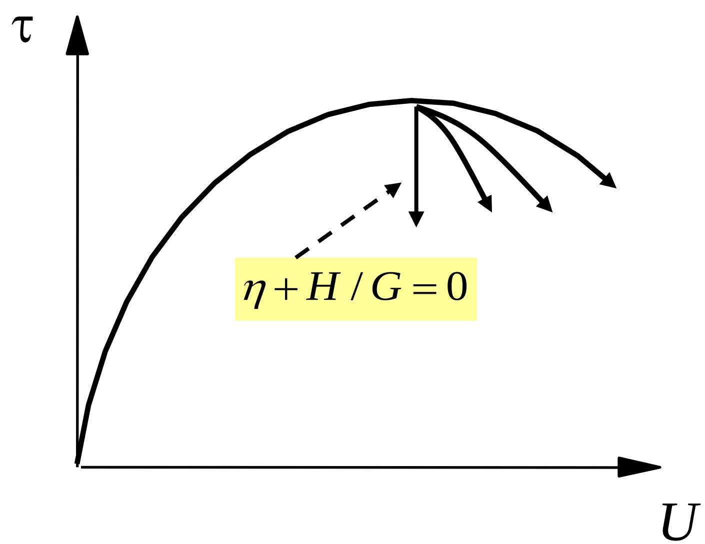
$$H_{\text{out}}^{\tan} = G \text{ (elastic unloading),}$$

$$H_{\text{band}}^{\tan} = H / (1 + H / G) \text{ (plastic loading)}$$

$$\dot{\gamma}^{\text{band}} = \frac{\dot{U}}{h} \frac{G}{\eta G + (1-\eta) H / (1 + H / G)}$$

$$\dot{\gamma}^{\text{out}} = \frac{\dot{U}}{h} \frac{H / (1 + H / G)}{\eta G + (1-\eta) H / (1 + H / G)}$$

$$\dot{\tau} = \frac{\dot{U}}{h} \frac{H}{\eta + H / G}$$



A quasi-static solution is not possible beyond the point where $H/G \leq -\eta$

1D Localization in a layer under simple shear

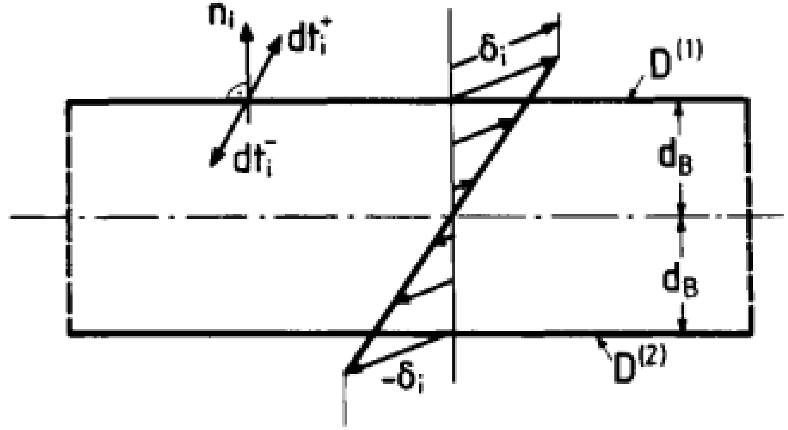
Summary:

- ✓ Non homogeneous deformation occurs past the peak of the shear stress vs shear strain curve.
- ✓ The non homogeneous solution is not unique. Different solutions with different shear band widths or with multiple shear bands exist.
- ✓ In the post-peak regime, deformations localize inside a shear band whereas elastic unloading occurs outside
- ✓ Past peak, the load vs displacement curve is not unique and depends upon the actual thickness of the strain localized zone.
- ✓ Information on the number and the width of the shear bands cannot be obtained with constitutive models without a characteristic length scale (e.g. lectures of P. Papanastasiou, A. Zervos, E. Papamichos, F. Collin tomorrow).

Deformation band model

kinematical compatibility condition
(weak discontinuity)

The continuity of the velocity field across the shear band implies that the tangential component of the velocity gradient is continuous across the band (Maxwell theorem)



$$\left[\Delta u_i\right] = 0 \text{ and } \left[\partial_j \Delta u_i\right] = g_i n_j$$

[.] jump across the shear band boundary

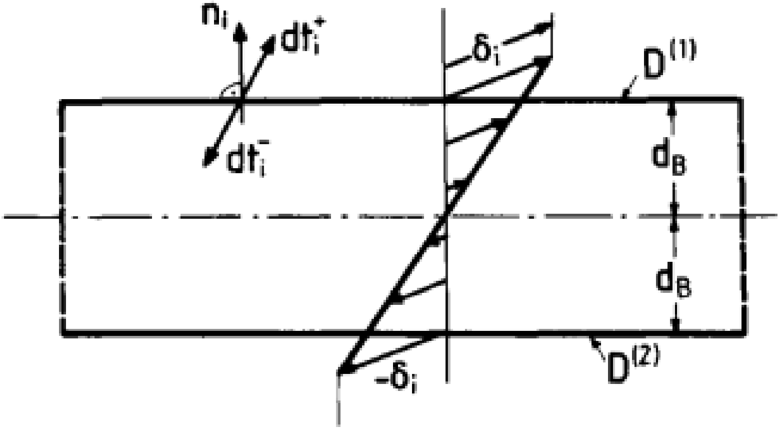
The above expression assumes that the non homogeneous solution has the form of a planar band.

Consequently, localization is favored when the pre-bifurcation, homogeneous field contains a plane of zero extension rates, as in plane strain whereas highly destabilizing effects as strong softening behaviour is needed to generate shear band formation in axisymmetric deformation.

Deformation band model

Equilibrium across the shear band boundary

$$\left[\Delta t_i\right] = \left[\Delta\sigma_{ij}\right].n_j = 0$$



Incremental constitutive relationships

$$\Delta\sigma_{ij} = C_{ijkl}\partial_l \Delta u_k$$

Two possibilities:

Discontinuous bifurcation: elastic unloading occurs outside the band while continued elastic-plastic loading occurs within the band.

Continuous bifurcation: The constitutive tensor is continuous across the band

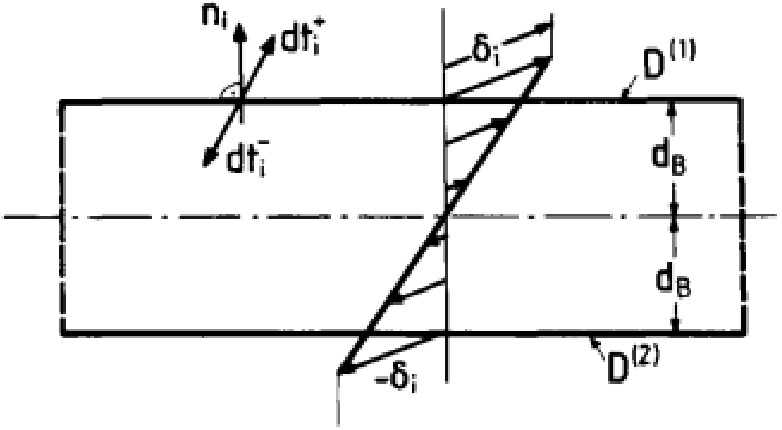
Continuous bifurcation precedes discontinuous bifurcation (Rice and Rudnicki 1980, Simo, 1993)

Deformation band model

$$\left[C_{ijkl} \right] = 0$$

$$\left[\Delta\sigma_{ij} \right].n_j = C_{ijkl} \left[\partial_l \Delta u_k \right].n_j$$

$$= C_{ijkl} n_j n_l g_k = 0$$



$$\Gamma_{ik} = C_{ijkl} n_j n_l$$ Acoustic tensor

$$\Gamma_{ik} g_k = 0$$

Condition of existence of a weak discontinuity for the incremental displacement

Strain localization criterion $\det \Gamma = 0$

Deformation band model

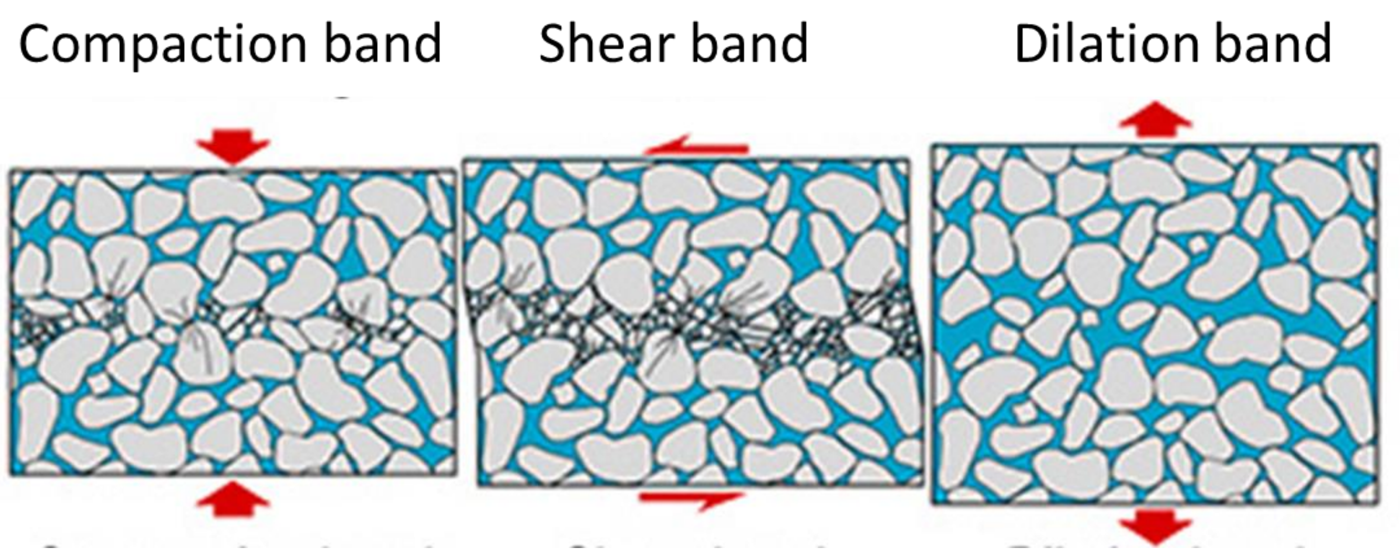
$\boldsymbol{n.g} = 1$: pure dilation band

$\boldsymbol{n.g} = -1$: pure compaction band

$\boldsymbol{n.g} = 0$: simple shear

$-1 < \boldsymbol{n.g} < 0$: compactive shear band

$0 < \boldsymbol{n.g} < 1$: dilatant shear band



Du Bernard et al., 2002

Deformation band model

Strain localization criterion $\det\Gamma = 0$ $\quad \Gamma_{ik} = C_{ijkl} n_j n_l$

Strong Ellipticity condition: $\forall n, C_{ijkl} n_j n_l$ strictly definite positive

• The *strain localization criterion* corresponds to the state of loss of ellipticity of the governing equations.

• They change type and from *elliptic* they turn to *hyperbolic*.

• Shear bands are thus identified with the *characteristic lines* of the governing hyperbolic partial differential equations.

*Wave propagation along the direction **n***

The wave velocity c is solution of : $\det\left(\Gamma_{ik} - \rho c^2 \delta_{ik}\right) = 0$

• If the acoustic tensor is strictly definite positive, all the velocities of acceleration waves are real (*Hadamard's (dynamic) stability criterion,* 1903).
• The localization criterion corresponds to a state for which the velocity of wave propagation in the direction normal to the band is null (*stationary wave*).

Strain localization and plasticity

Example of a constitutive model commonly used for rocks:

Drucker-Prager plasticity model with non associate flow rule:

$$F = \overline{\tau} - \mu(q - \sigma); Q = \overline{\tau} + \beta\sigma$$

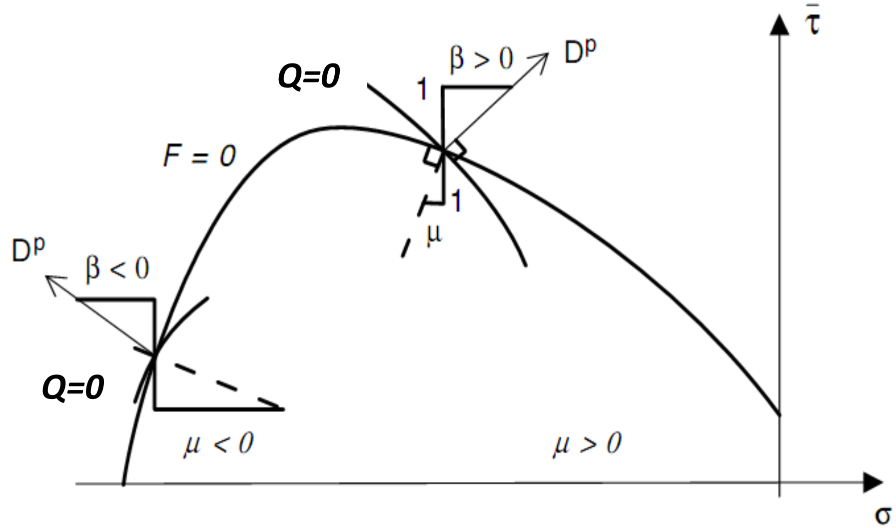
Mean stress: $\sigma = \sigma_{kk} / 3$

Mises equivalent stress: $\overline{\tau} = \sqrt{s_{ij} s_{ij} / 2}$,

with $s_{ij} = \sigma_{ij} - \sigma\delta_{ij}$

friction coefficient: μ

dilatancy coefficient: β

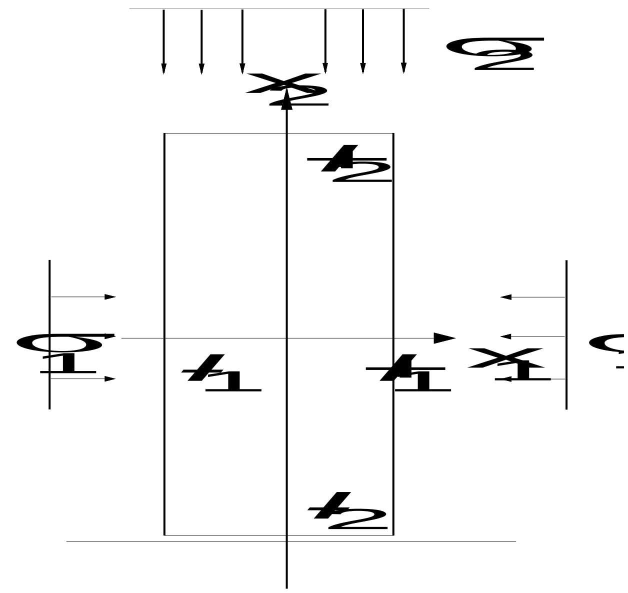


For low mean stress, the behavior is *frictional and dilatant*, μ and β are positive.

For higher mean stress, the behavior is *compacting*, μ and β are negative.

EXAMPLE: 2D COMPRESSION TEST

(small strain analysis)



Elasto-plastic incremental constitutive equations

$$\dot{\sigma}_{11} = L_{11}\dot{\varepsilon}_{11} + L_{12}\dot{\varepsilon}_{22}$$
$$\dot{\sigma}_{22} = L_{21}\dot{\varepsilon}_{11} + L_{22}\dot{\varepsilon}_{22}$$
$$\dot{\sigma}_{12} = 2G\dot{\varepsilon}_{12}$$

$$L_{11} = G\left(1+\kappa - \frac{1}{H}(1+\kappa\mu)(1+\kappa\beta)\right)$$
$$L_{12} = G\left(-1+\kappa - \frac{1}{H}(1+\kappa\mu)(-1+\kappa\beta)\right)$$
$$L_{21} = G\left(-1+\kappa - \frac{1}{H}(-1+\kappa\mu)(1+\kappa\beta)\right)$$
$$L_{22} = G\left(1+\kappa - \frac{1}{H}(-1+\kappa\mu)(-1+\kappa\beta)\right)$$

μ: friction coefficient, β dilatancy coefficient

$\kappa = K/G = 1/(1-2\nu)$ (ν is the Poisson's ratio),

$H = 1 + h + \kappa\mu\beta$: plastic modulus,

$h = |\sigma| \dfrac{d\mu}{d\gamma^p}$: hardening modulus

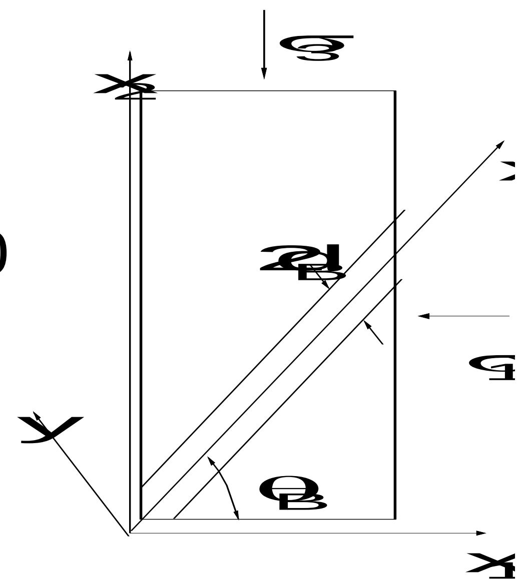
2D COMPRESSION TEST

$$\det(\Gamma_{ij}) = 0$$

$$\boldsymbol{\Gamma} = \begin{pmatrix} L_{11}n_1^2 + Gn_2^2 & \left(L_{12}+G\right)n_1n_2 \\ \left(L_{21}+G\right)n_1n_2 & L_{22}n_2^2 + Gn_1^2 \end{pmatrix}$$

$$\det(\boldsymbol{\Gamma}) = GL_{11}n_1^4 + \left(L_{11}L_{22} - L_{12}L_{21} - GL_{12} - GL_{21}\right)n_1^2n_2^2 + GL_{22}n_2^4$$

Shear-band inclination angle θ $\qquad \tan\theta = -(n_1 / n_2)$



Characteristic equation: $\qquad a\tan^4\theta + b\tan^2\theta + c = 0$

$$a = L_{11}^*; \quad b = L_{11}^*L_{22}^* - L_{12}^*L_{21}^* - L_{12}^* - L_{21}^*; c = L_{22}^*$$

$$\text{for } i = 1,2 \quad L_{ij}^* = L_{ij} / G$$

2D COMPRESSION TEST

Classification of the regimes of the characteristic equation

Elliptic imaginary (EI)	4 imaginary roots	$\Delta = b^2 - 4ac \geq 0$ $b/a > 0$ et $c/a \geq 0$
Elliptic complex (EC)	4 complex roots	$\Delta < 0$
Parabolic (P)	2 real roots 2 imaginary roots	$\Delta = b^2 - 4ac > 0$ $c/a < 0$
Hyperbolic (H)	4 real roots	$\Delta = b^2 - 4ac \geq 0$ $b/a \leq 0$ et $c/a > 0$

2D COMPRESSION TEST

The condition for shear band formation is derived from the requirement that the characteristic equation has real solutions.

This condition is firstly met at a state C_B (B for bifurcation) for which

$$C_B : b/a < 0 \quad \text{and} \quad D = b^2 - 4ac = 0$$

$$\theta_B = \pm \arctan\left(c^{1/4}\right)$$

critical hardening modulus h_B at shear banding

$$h_B = \frac{(\mu - \beta)^2}{8(1-\nu)}$$

-For associate plasticity ($\mu=\beta$) localisation occurs at peak (h_B=0)

-For non associate plasticity ($\beta<\mu$) localisation occurs in the hardening regime

$$\theta_B \approx \frac{\pi}{4} + \frac{\phi_B}{4} + \frac{\psi_B}{4}$$

Arthur *et al*. 1977, Vardoulakis, 1980

For associate plasticity ($\phi_B=\psi_B$) the Coulomb orientation is retrieved:

$$\theta_B = \pi/4+\phi/2$$

Shear band formation in rocks
General 3D state of stress

Critical hardening modulus at shear banding (Rudnicki and Rice, 1975)

$$h_B = \frac{1+\nu}{9(1-\nu)}(\beta-\mu)^2 - \frac{1+\nu}{2}\left(N + \frac{\beta+\mu}{3}\right)^2$$

$N = \dfrac{s_{II}}{\bar{\sigma}}$ is the Intermediate normalized principal deviatoric stress

$-1/\sqrt{3} \le N \le 1/\sqrt{3}$

for axisymmetric extension

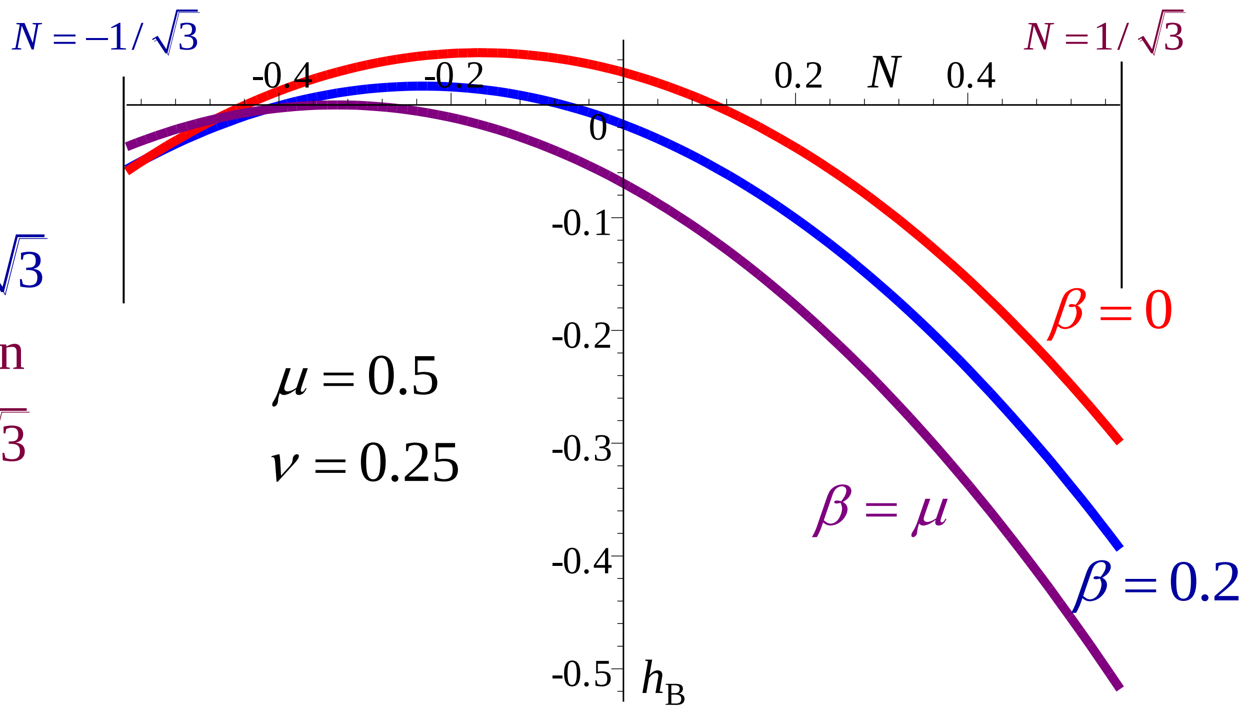
$(\sigma_1 = \sigma_2 < \sigma_3 < 0)$, $N = -1/\sqrt{3}$

for axisymmetric compression

$(\sigma_3 < \sigma_1 = \sigma_2 < 0)$, $N = 1/\sqrt{3}$

for pure shear

$(\sigma_1 = -\sigma_3, \sigma_2 = 0)$, $N = 0$

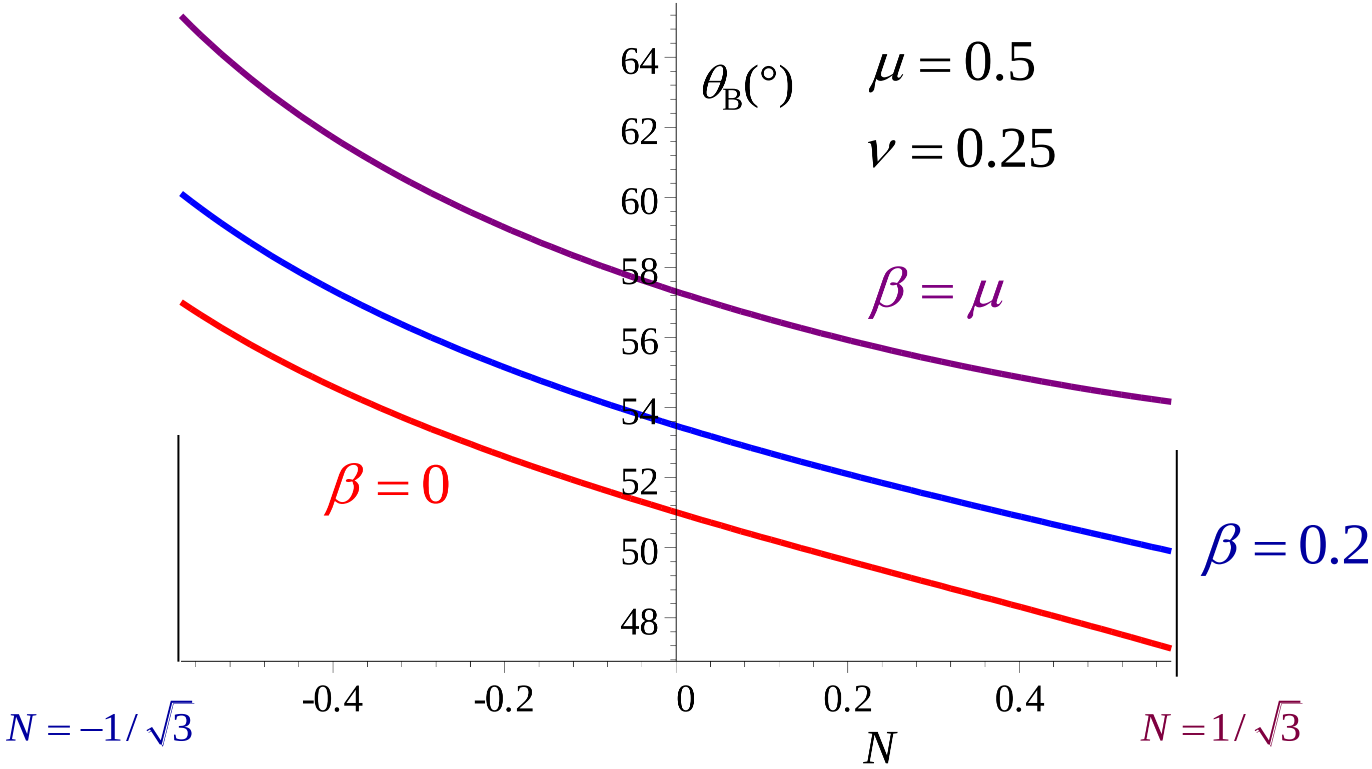


Shear band formation in rocks

General 3D state of stress

Shear band orientation with respect to the least (in absolute value) compressive stress (Rudnicki and Olsson, 1998)

$$\theta_B = \frac{\pi}{4} + \frac{1}{2}\arcsin\alpha, \text{ with } \alpha = \frac{(2/3)(1+\nu)(\beta+\mu) - N(1-2\nu)}{\sqrt{4-3N^2}}$$



Shear band formation in rocks

Dilation and compaction bands

$$\theta_B = \frac{\pi}{4} + \frac{1}{2}\arcsin\alpha, \text{ with } \alpha = \frac{(2/3)(1+\nu)(\beta+\mu) - N(1-2\nu)}{\sqrt{4-3N^2}}$$

This expression is valid for -1 ≤ α ≤ 1 (Perrin and Leblond, 1993)

$$\frac{N(1-2\nu) - \sqrt{4-3N^2}}{2(1+\nu)/3} \le \beta + \mu \le \frac{N(1-2\nu) + \sqrt{4-3N^2}}{2(1+\nu)/3}$$

$$\text{For } \beta + \mu > \frac{N(1-2\nu) + \sqrt{4-3N^2}}{2(1+\nu)/3}, \text{ formation of dilation bands, } \theta_B = 90^\circ$$

$$\text{For } \beta + \mu < \frac{N(1-2\nu) + \sqrt{4-3N^2}}{2(1+\nu)/3}, \text{ formation of compaction bands, } \theta_B = 0^\circ$$

Shear band formation in element tests on rocks

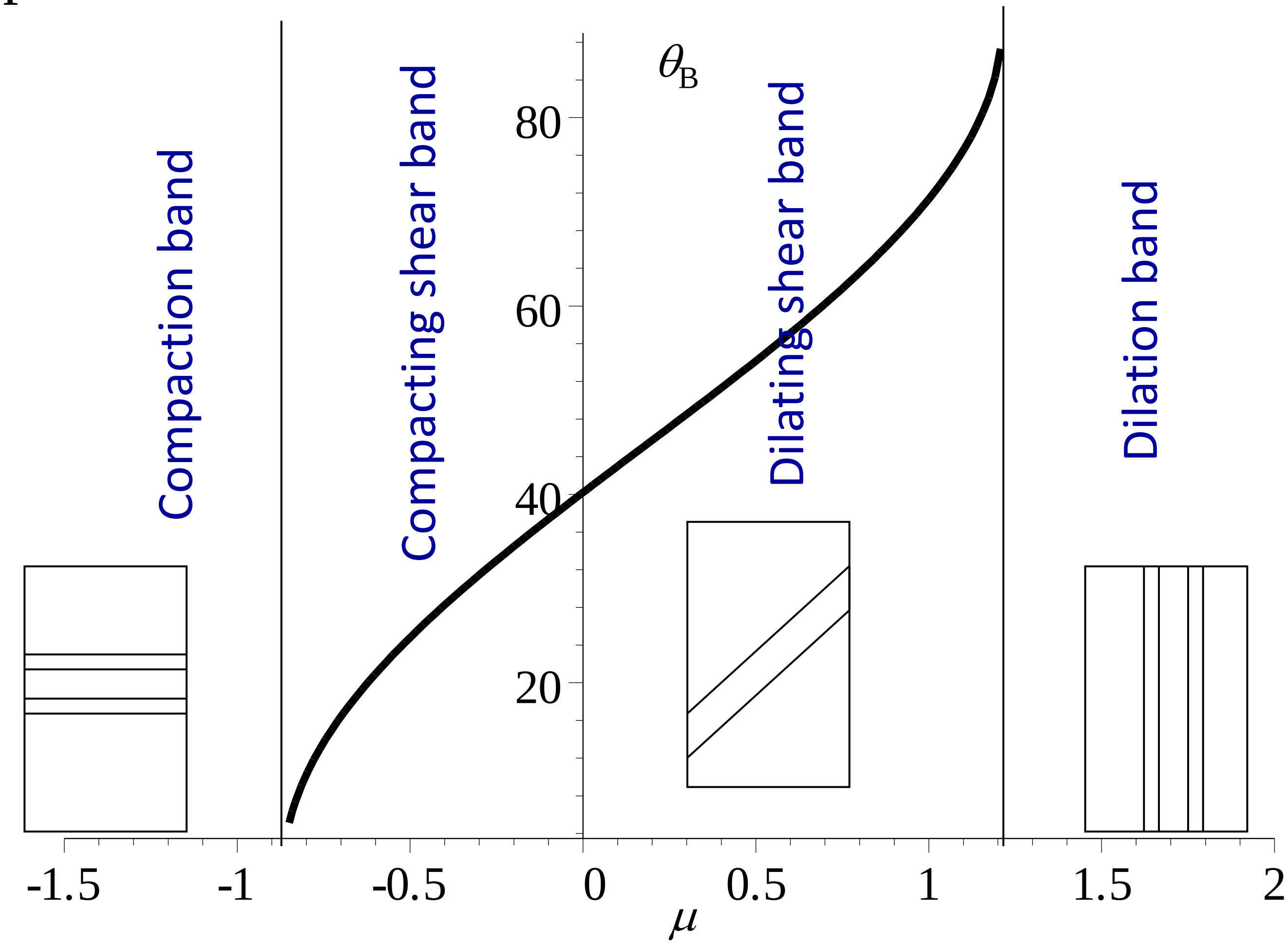
Dilation and compaction bands

axisymmetric compression

$N = 1/\sqrt{3}$

$\beta = \mu$

$\nu = 0.25$

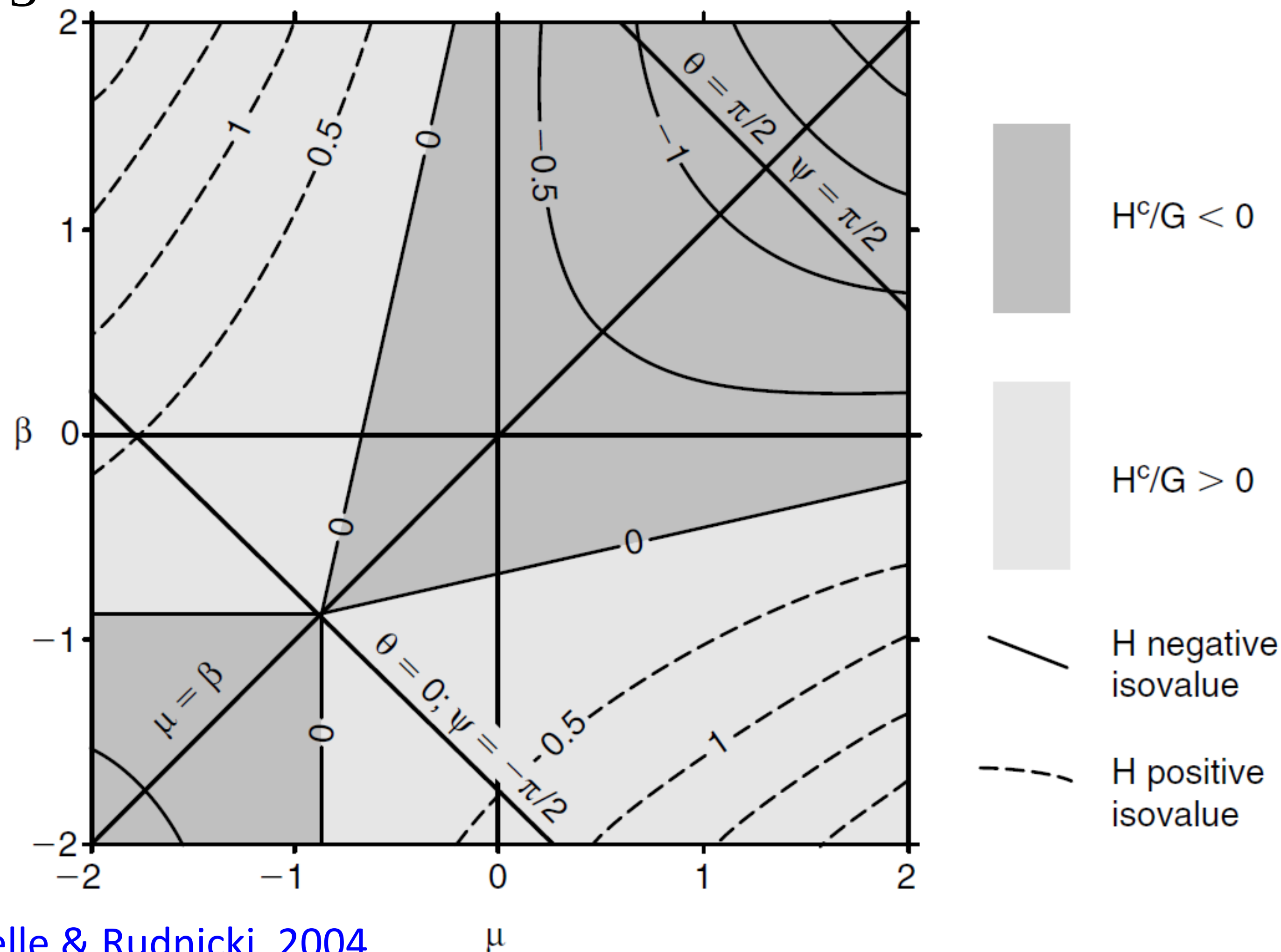


Shear band formation in element tests on rocks

Dilation and compaction bands

axisymmetric compression

$$N = 1/\sqrt{3}$$

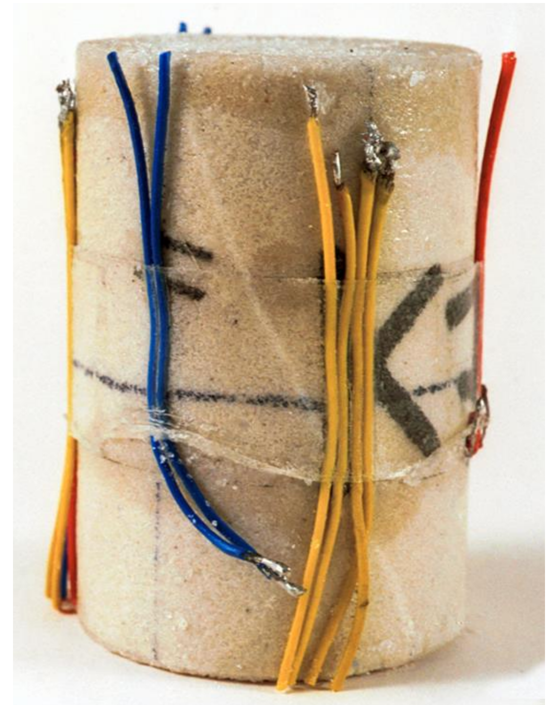


from Bésuelle & Rudnicki, 2004

Conjugate shear bands in perlite

(Milos Island, Greece)

(photo I. Vardoulakis)

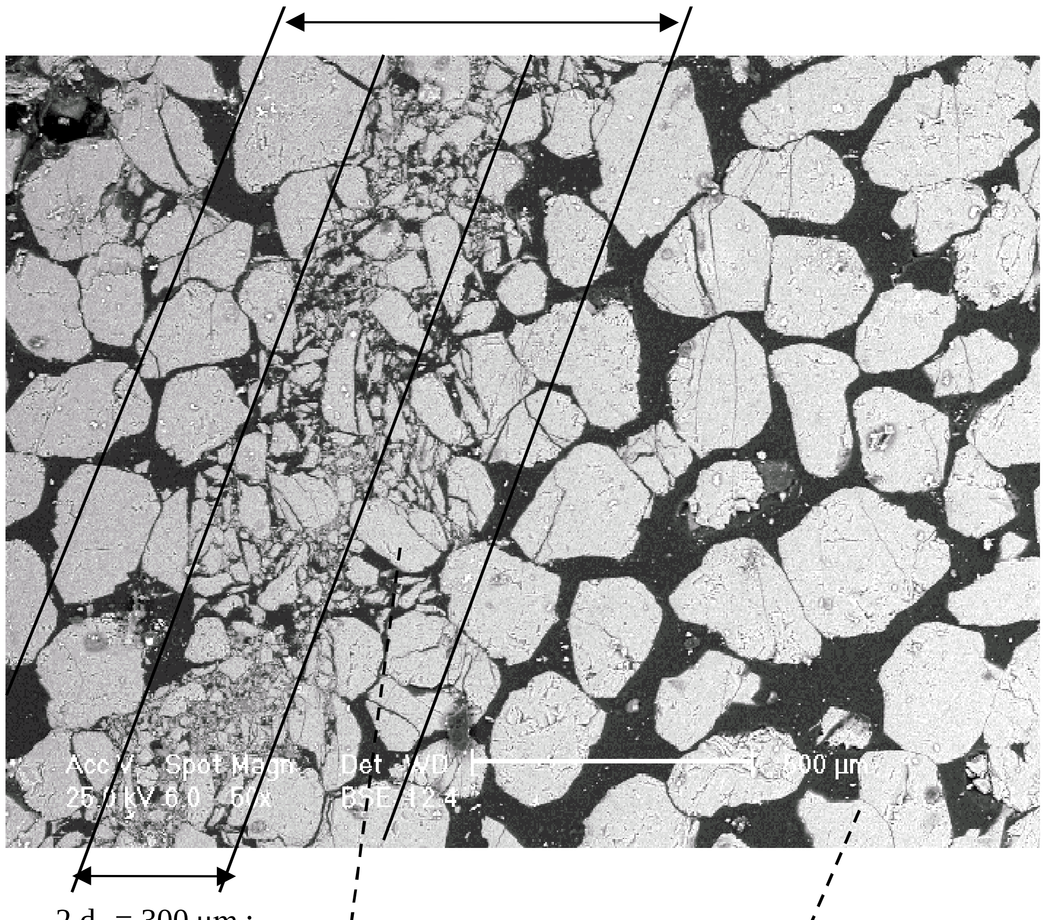


Shear band formed in triaxial test on Fontainebleau sandstone

El Bied and Sulem (2002)

Shear band formed in triaxial testing on Fontainebleau sandstone (porosity 21%, grain size 0.2mm)

2 d_B = 660 µm : thickness of the shear band as measured with the magnifying glass



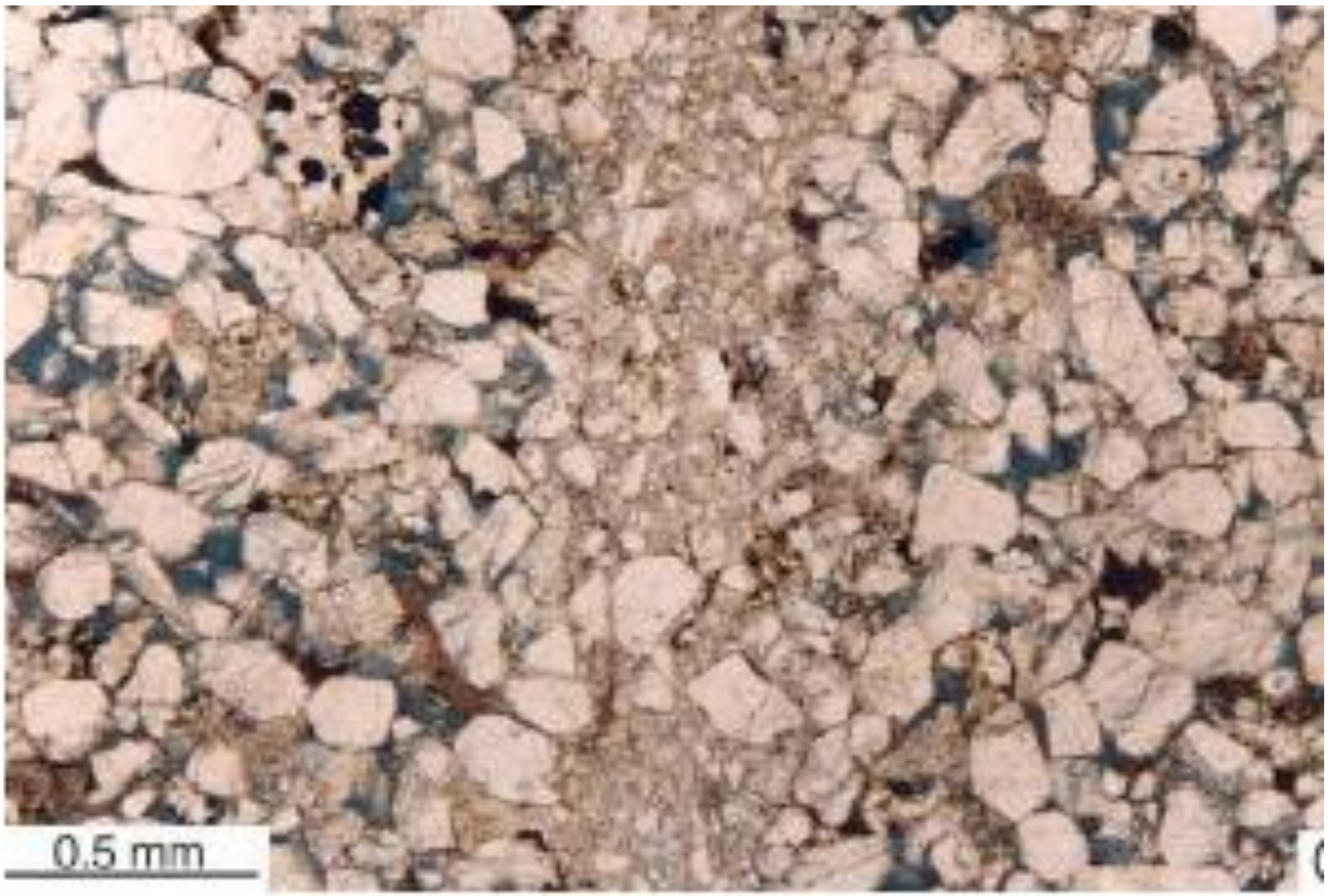
2 d_C = 300 µm : thickness of the crushed zone

cracked zone

intact zone

Elbied and Sulem, 2002, *Int. J. Rock Mech Min Sci*

P. Labaume, I. Moretti / Journal of Structural Geology 23 (2001) 1659–1675

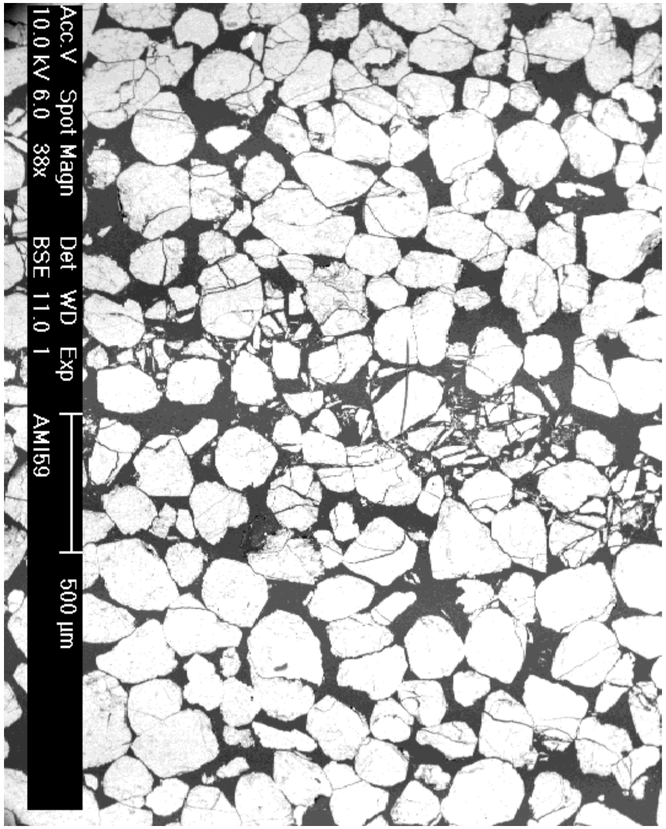


Fault zone in porous sandstone

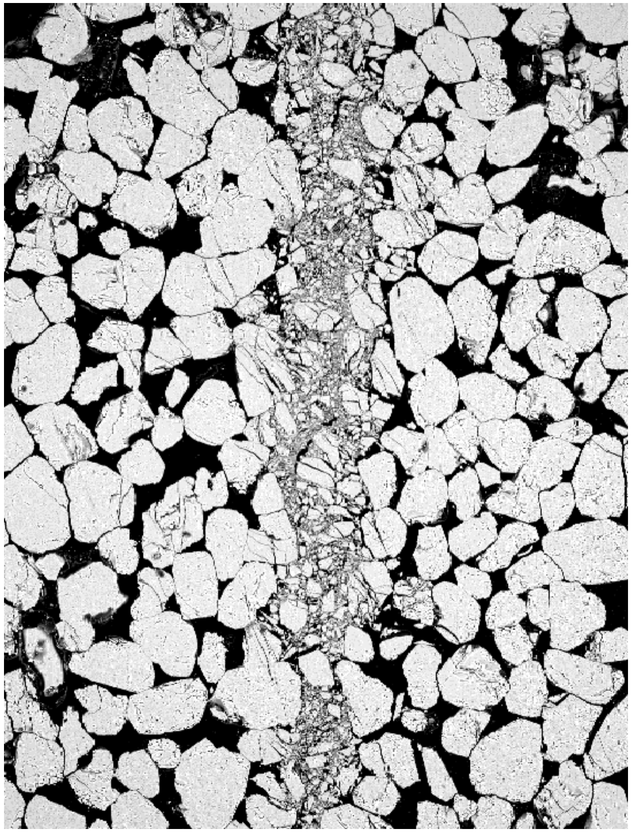
Shear band formed in triaxial testing on Fontainebleau sandstone (porosity 21%, grain size 0.2mm)

Cataclastic shear banding

Sulem & Ouffroukh, 2006, Int. J. Rock Mech. Min. Sci



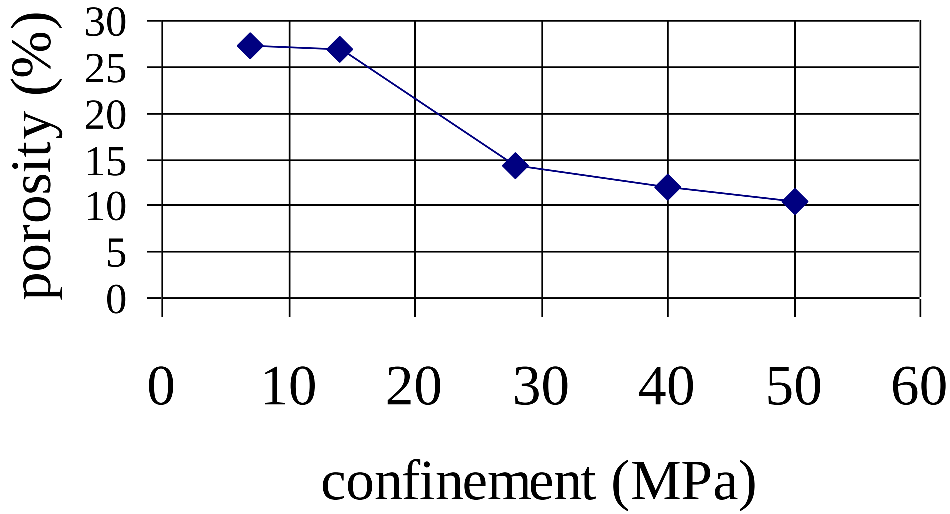
Confining pressure: 7 MPa

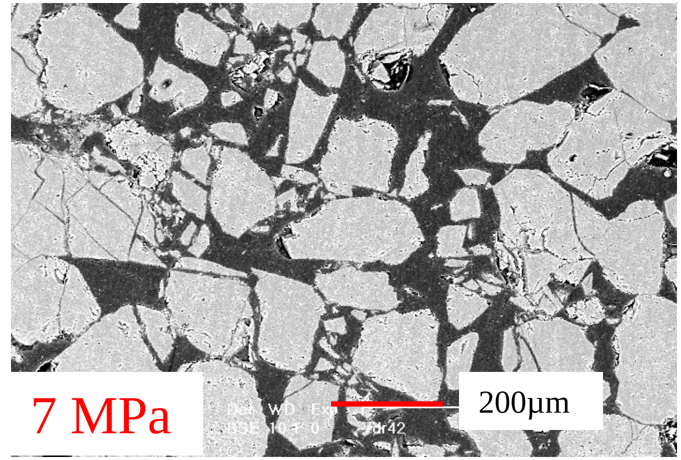


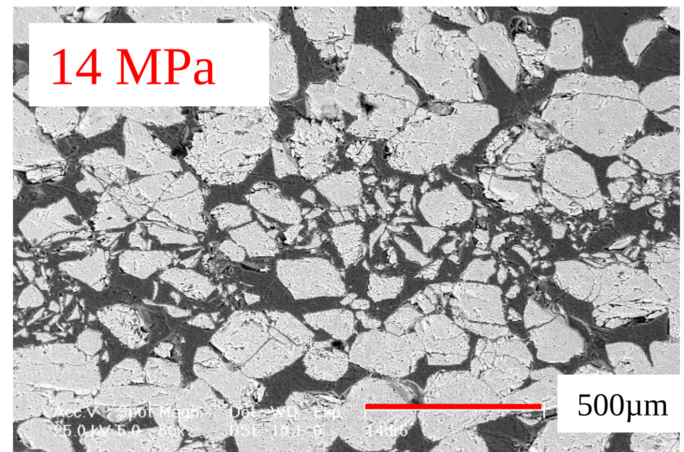
Confining pressure : 28 MPa

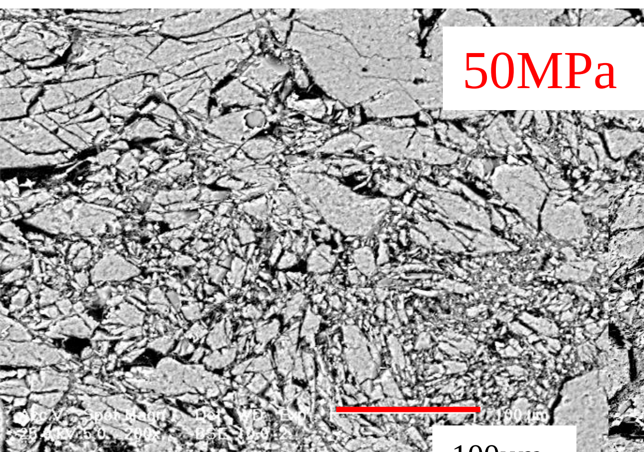
Porosity inside the shear band

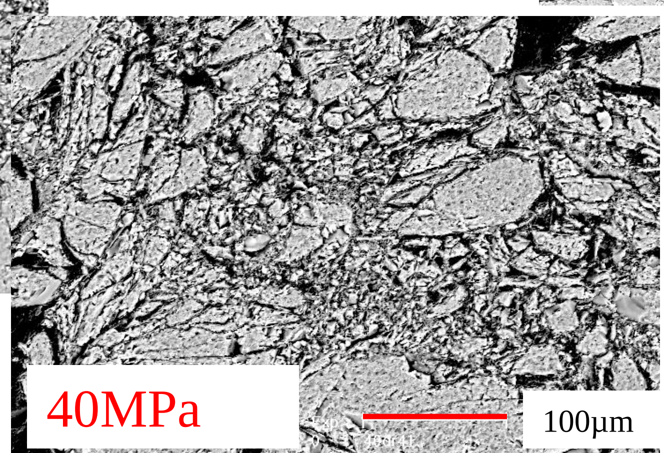
Drained tests

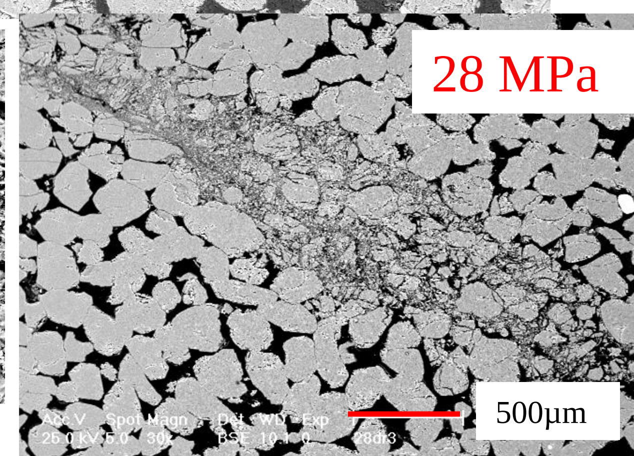






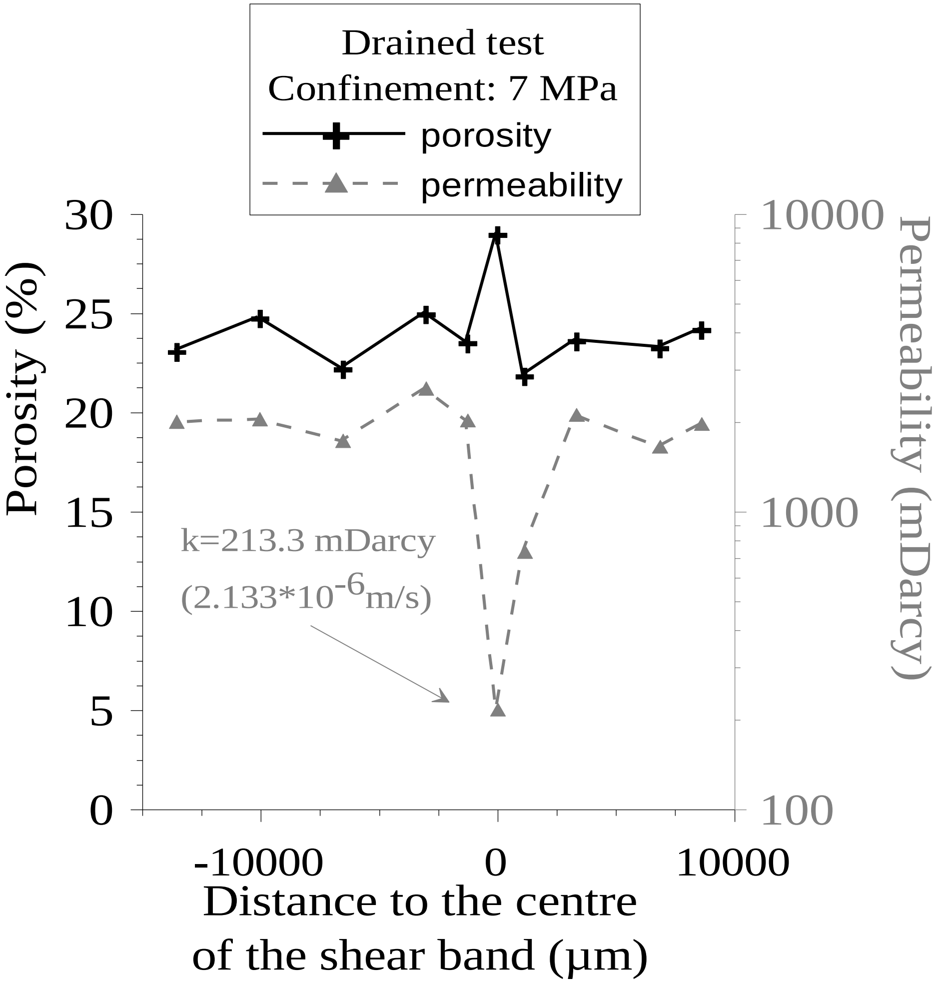


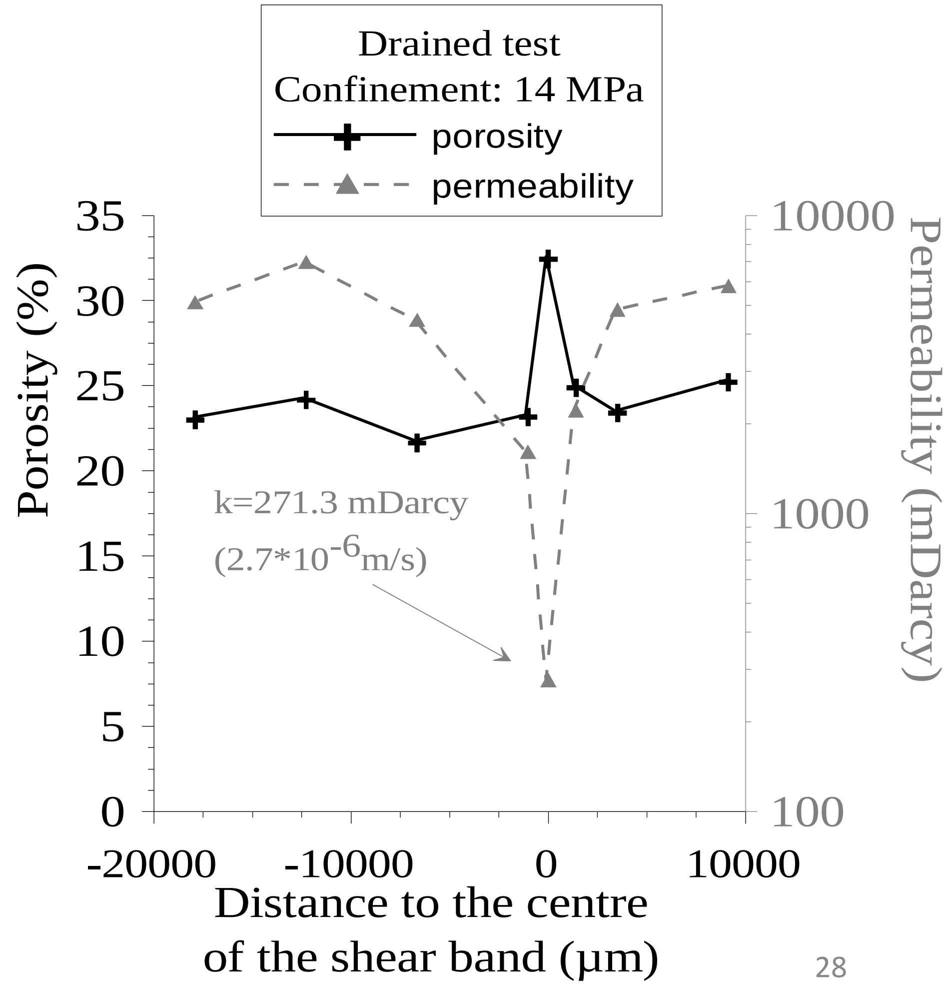




At low confining pressure:

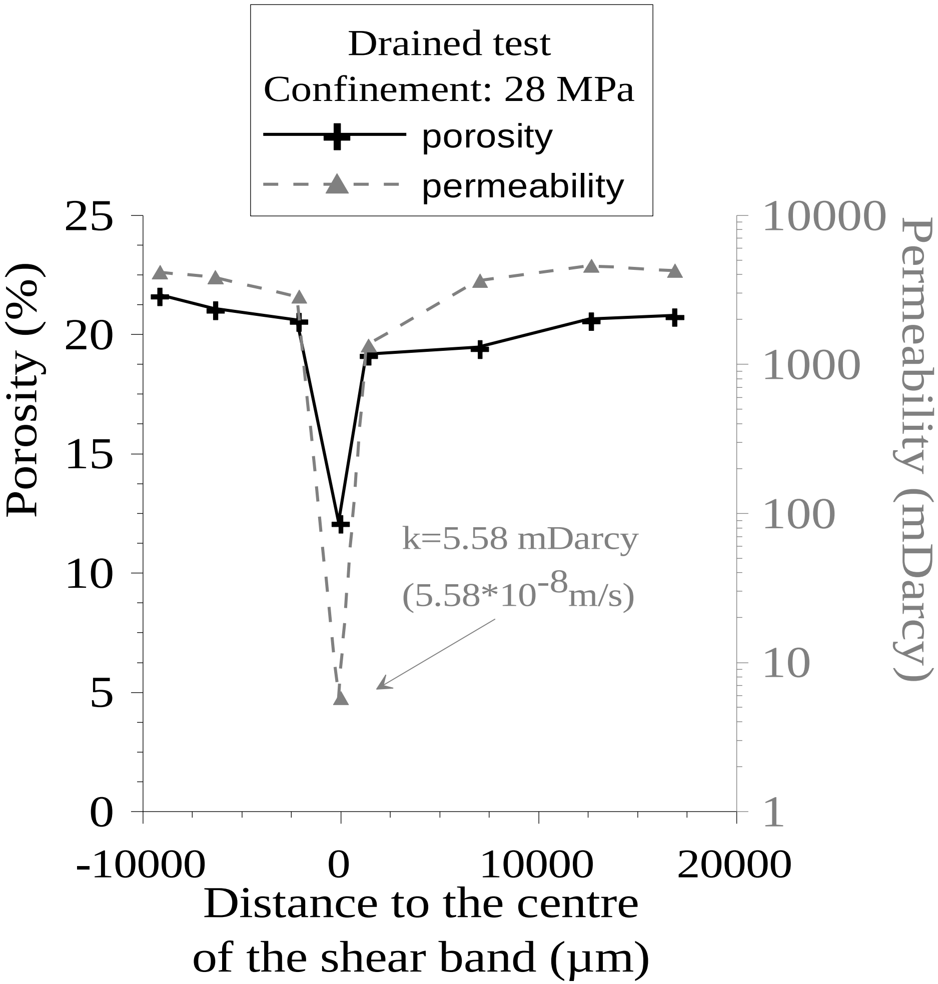
porosity **increase** and permeability **decrease**

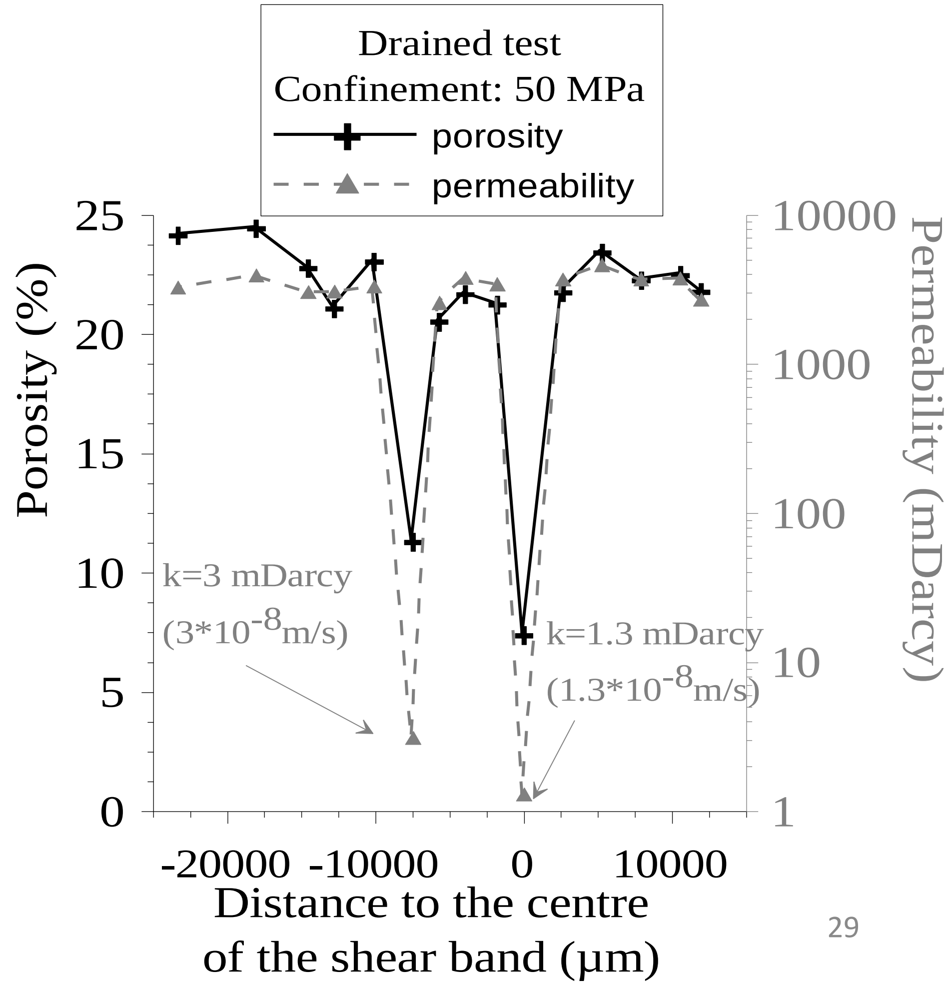




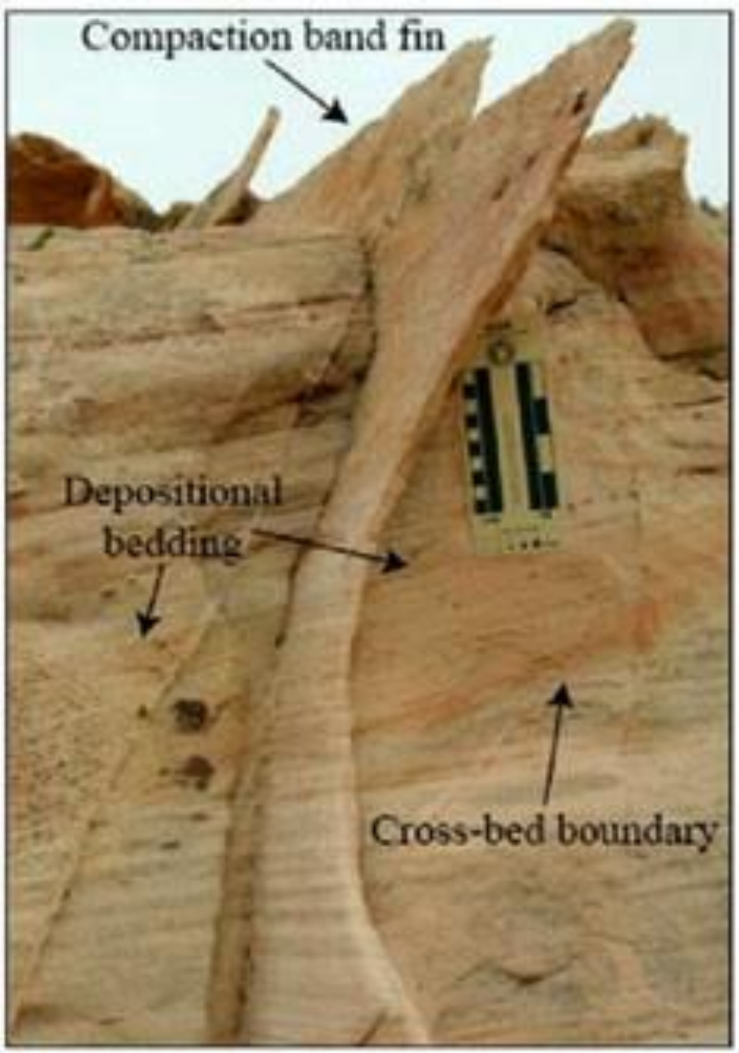
At high confining pressure:

porosity **decrease** and important permeability **decrease**

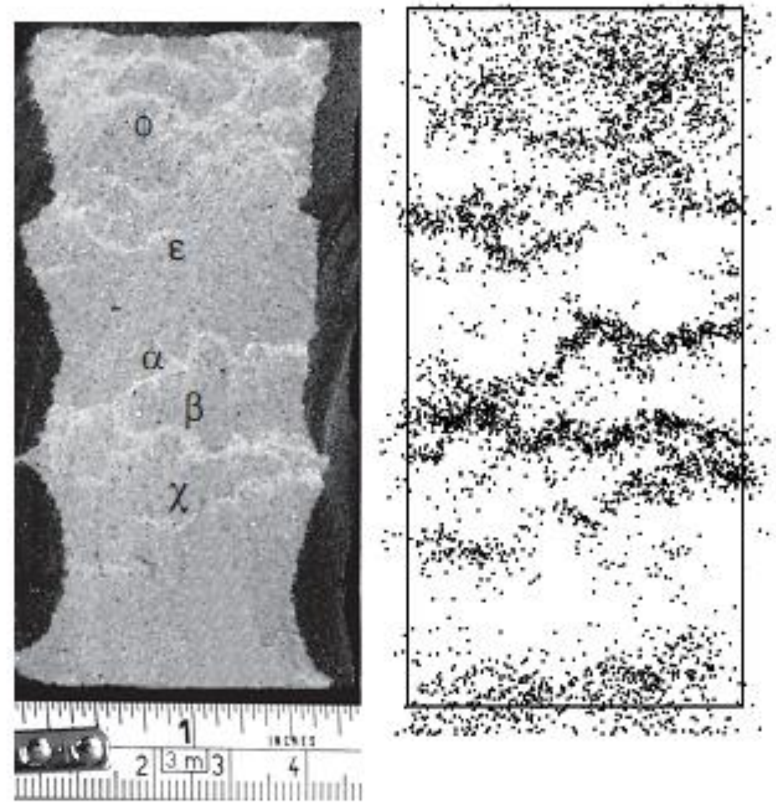




Compaction bands



Compaction bands in sandstone East Nevada, (Sternlof, 2006)



Compaction bands in sandstone Triaxial compression, (Fortin *et al*., 2005)

Non coaxial plasticity

Classical plasticity flow rule: $$\dot{\varepsilon}_{ij}^{p} = \dot{\lambda}\frac{\partial Q}{\partial \sigma_{ij}}$$

• The direction of the plastic deformation is fixed by the current state of stress and does not depend upon the direction of the stress increment.

• The plastic deformation rate possesses the same principal axes as the stress tensor, which means that it is *coaxial* to the stress tensor.

• For better predictions for shear-band formation, one has to consider non coaxial plasticity flow rule or resort to *hypoplasticity* flow rules which consider the effect of stress rate.

Examples:
- Yield vertex plasticity model (Rudnicki & Rice, 1975)
- Deformation theory of plasticity (Vermeer and Schotman, 1986, Sulem & Vardoulakis, 1990)
- Non coaxial plasticity model (Papamichos and Vardoulakis, 1995)
- Incrementally non-linear laws (Darve, 1985, Chambon and Desrues, 1989)

Post bifurcation behavior
Limits of the classical continuum theory

- In the post bifurcation regime the governing equations are mathematically *ill-posed* (loss of uniqueness, Hadamard sense)

- Conventional constitutive models do not have an *internal length* (material parameter with the dimension of length), so that the shear band thickness (i.e. the extent of the plastically softening region) is undetermined.

- It appears necessary to resort to continuum models with *microstructure* to describe correctly localization phenomena.

Post bifurcation behavior
Limits of the classical continuum theory

•These generalized continua usually contain additional kinematical degrees of freedom (*Cosserat, higher grade continuua*, lectures of E. Papamichos, P. Papanastasiou, A. Zervos, F. Collin).

• In this case the underlying mathematical problem describing localization phenomena is *regularized* and the governing equations remain elliptic.

• Moreover, this technique allows *robust computations* to follow the evolution of the considered system in the post-bifurcation regime and to extract additional information such as the *shear band thickness* or to assess *the scale effect*.

Strain localization in fluid saturated porous media

Constitutive equations (Benallal & Comi, 2003, Coussy, 2004)

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}^{d}\dot{\boldsymbol{\varepsilon}} - \mathbf{K}\dot{p}, \quad p \text{ is the pore pressure}$$

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}^{u}\dot{\boldsymbol{\varepsilon}} - \frac{1}{D}\mathbf{K}\dot{\zeta}, \quad \zeta \text{ is the fluid mass content}$$

Stability of homogeneous deformation

Perturbation field: $\delta\mathbf{X} = \tilde{\mathbf{X}}\exp\left(i\xi\mathbf{n.x} + st\right)$

Instability in the form of unbounded growth of the perturbation occurs when:

$$sD\det\left[\mathbf{n.C}^{u}\mathbf{.n}\right] + k\xi^{2}\det\left[\mathbf{n.C}^{d}\mathbf{.n}\right] = 0$$

k is the permeability of the medium

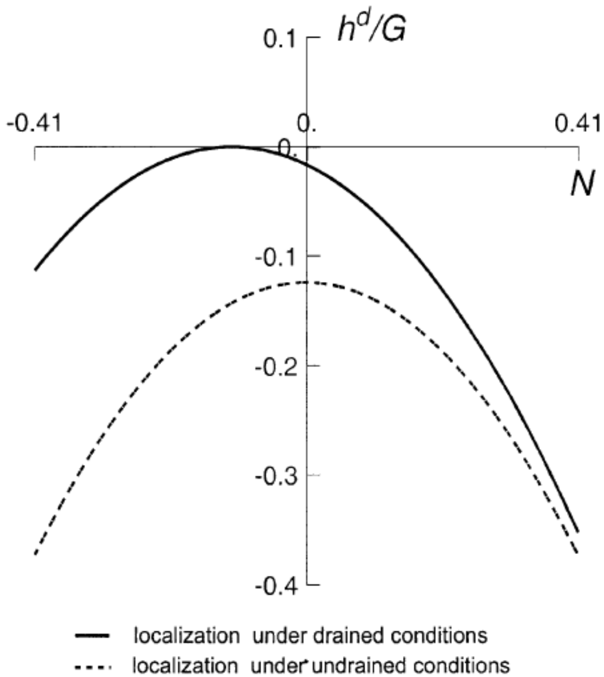
Strain localization in fluid saturated porous media

Strain localization occurs when the drained or the undrained acoustic tensor becomes singular (Rudnicki, 2000)

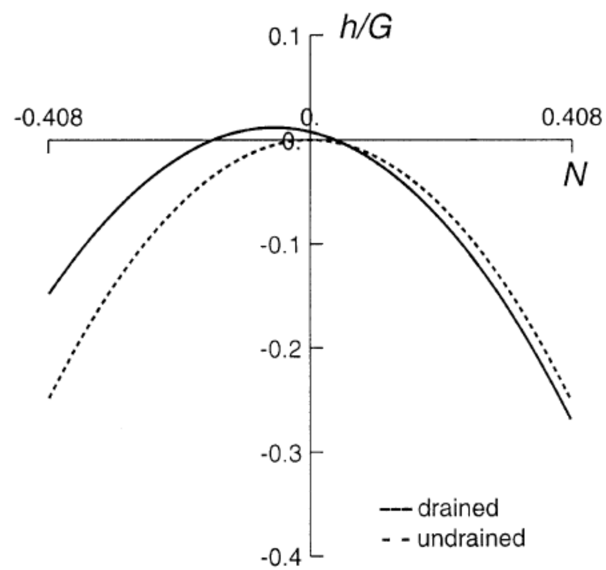
$$\det\left[\mathbf{n.C}^{d}\mathbf{.n}\right]=0 \quad \text{or} \quad \det\left[\mathbf{n.C}^{u}\mathbf{.n}\right]=0$$

- ✓ The two conditions must be checked on the real deformation path (which is not necessarily drained or undrained.) to infer which one is met first.
- ✓ For associative behavior, it is shown that the singularity of the drained acoustic tensor occurs before the singularity of the undrained acoustic tensor which means that instability occurs when the condition of localization of the underlying drained deformation is met (Rice 1975, Benallal & Comi, 2000)
- ✓ For non-associative behavior, instability is controlled either by the drained or the undrained properties, depending on the constitutive equations and on the loading path.
- ✓ For compactant behaviour as for highly porous rocks, conditions of localization can be met for undrained response before they are met in terms of the drained response.

Strain localization in fluid saturated porous media



(a)



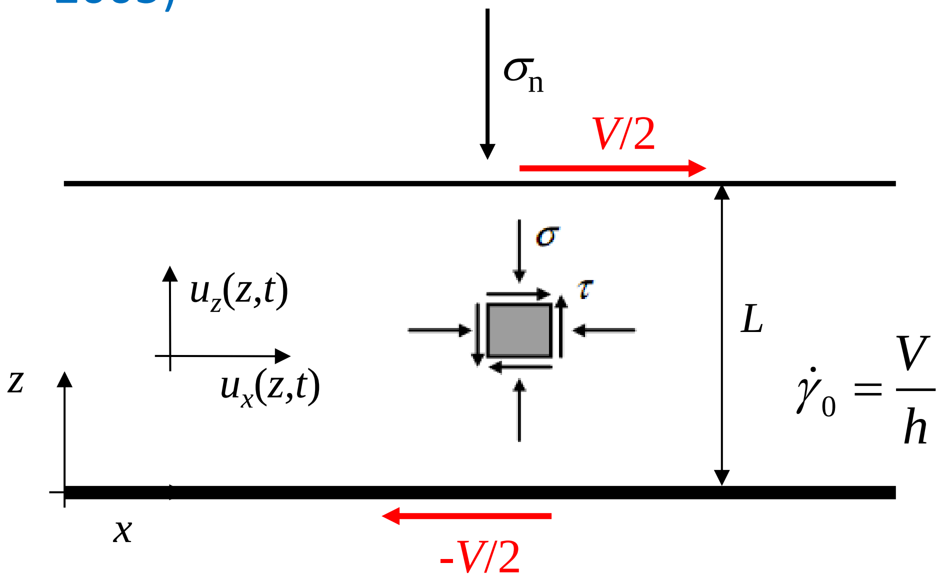
(b)

$$(E = 60 \text{ MPa},\ \nu = 0.3,\ M = 7500 \text{ MPa},\ b = 1,\ \mu = 0.08,\ \beta = 0)$$

Critical hardening modulus at localization under drained and undrained conditions for (a) associative flow rule, (b) non associative flow rule (after Benallal & Comi, 2003)

Example of a saturated layer sheared in plane strain under globally undrained conditions

(Rice, 1975, Vardoulakis, 1985, 1996, Rudnicki & Garagash, 2000, Garagash, 2005)



Shear strain and volume strain

$$\gamma = \frac{\partial u_x}{\partial z} \qquad \varepsilon = \frac{\partial u_z}{\partial z}$$

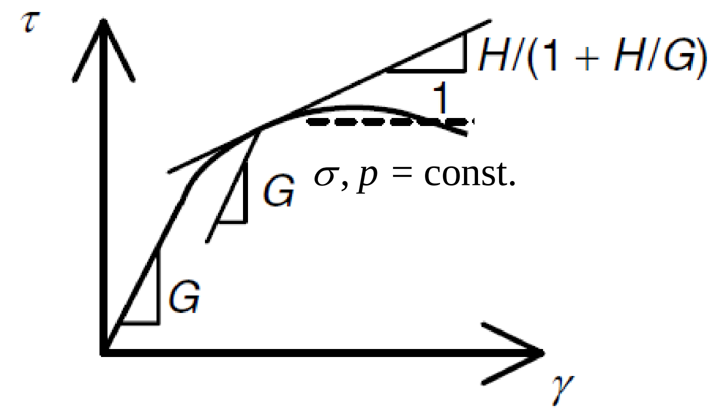
Uniform state of stress in the layer

$$\frac{\partial \tau}{\partial z} = 0 \qquad \frac{\partial \sigma}{\partial z} = 0$$

Elasto-plastic constitutive equations

$$d\gamma = \frac{1}{G} d\tau + d\gamma^p \; ; d\varepsilon = \frac{1}{M} d\left(\sigma - p\right) + d\varepsilon^p$$

$$d\gamma^p = \frac{1}{H}\left(d\tau - \mu d\left(\sigma - p\right)\right); d\varepsilon^p = \beta d\gamma^p$$



Fluid mass balance

Fluid mass per unit volume of porous medium $m_f = \rho_f n$

n is the pore volume fraction (Lagrangian porosity)

ρ_f is the density of the saturating fluid.

$$\frac{\partial m_f}{\partial t} = -\frac{\partial q_f}{\partial z}$$

q_f is the fluid flux

$$q_f = -\frac{\rho_f}{\eta_f} k_f \frac{\partial p}{\partial z}$$

Darcy law for the fluid flow, with viscosity η_f through a material with permeability k_f

For incompressible fluid and solid phase

$$\kappa \frac{\partial^2 p}{\partial z^2} = \frac{\partial \varepsilon}{\partial t}$$

$\kappa = k_f / (\rho_f \eta_f)$ is the permeability

Summary of the governing equations:

Equilibrium equations:

$$\frac{\partial \tau}{\partial z} = 0 \quad \frac{\partial \sigma}{\partial z} = 0$$

Constitutive equations:

$$d\gamma = \frac{1}{G} d\tau + \frac{1}{H}\left(d\tau - \mu d\left(\sigma - p\right)\right)$$

$$d\varepsilon = \frac{1}{M} d\left(\sigma - p\right) + \frac{\beta}{H}\left(d\tau - \mu d\left(\sigma - p\right)\right)$$

Fluid mass balance:

$$\kappa \frac{\partial^2 p}{\partial z^2} = \frac{\partial \varepsilon}{\partial t}$$

Boundary conditions:

$$\text{drained b.c.:} \;\; p(z = \pm L/2, t) = p_0$$

$$\text{undrained b.c.:} \;\; \frac{\partial p}{\partial z}(z = \pm L/2, t) = 0$$

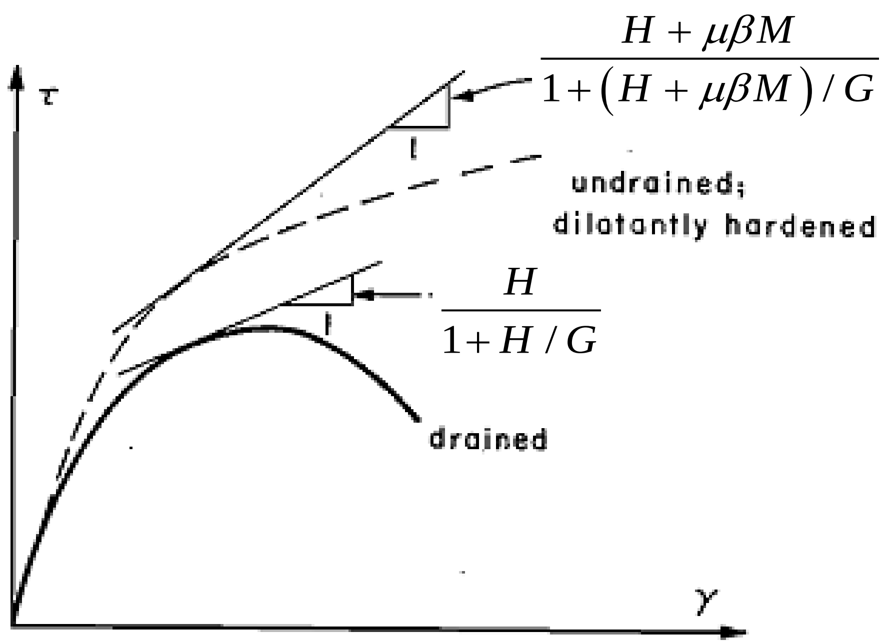
Homogeneous drained response:

$$dp = 0 \qquad d\gamma = \left(\frac{1}{G} + \frac{1}{H}\right) d\tau \text{ or } d\tau = \frac{H}{1 + H/G} d\gamma \qquad d\tau = h d\gamma$$

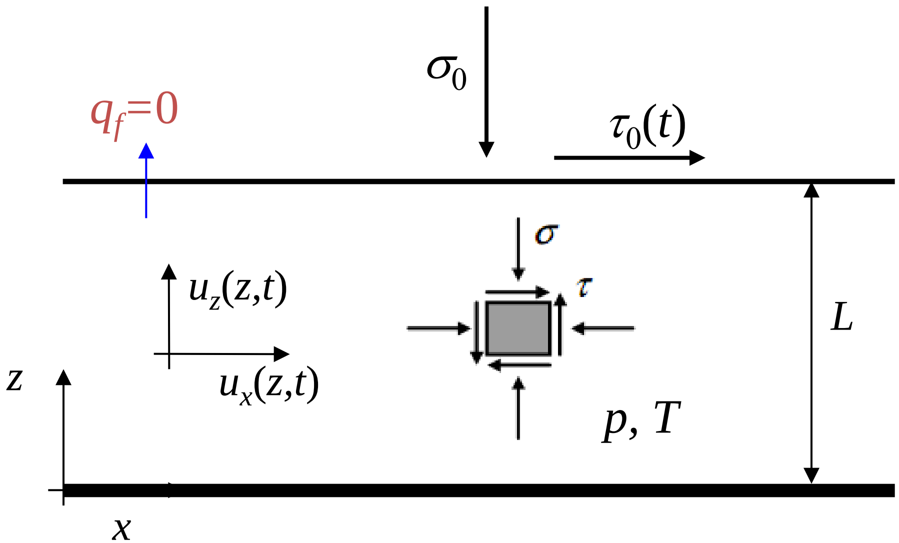
Homogeneous undrained response:

$$d\varepsilon = 0 \ \Rightarrow d\left(\sigma - p\right) = \frac{\beta M}{H + \beta\mu M} d\tau$$

$$d\gamma = \frac{1}{G} d\tau + \frac{1}{H}\left(d\tau - \mu d\left(\sigma - p\right)\right) \ \Rightarrow d\tau = \frac{H + \beta\mu M}{1 + \left(H + \beta\mu M\right)/G} d\gamma \qquad d\tau = h_u d\gamma$$



Stability analysis of undrained deformation



Fluid flux is prevented at the boundaries. However, internal fluid is permitted inside the layer.

Homogeneous solution:

$$\tau_0, \gamma_0, \varepsilon_0, p_0$$

Introduction of small perturbation quantities: $\tilde{\gamma}, \tilde{\tau}$ etc...

$$\gamma = \gamma_0 + \tilde{\gamma}\,; \varepsilon = \varepsilon_0 + \tilde{\varepsilon}\,; \sigma = \sigma_0 + \tilde{\sigma}\,; \tau = \tau_0 + \tilde{\tau}\,; p = p_0 + \tilde{p}$$

Physically, perturbations may correspond to either material imperfections or disturbances in the loading system

Equilibrium equations:

$$\frac{\partial \tau}{\partial z} = 0, \quad \frac{\partial \sigma}{\partial z} = 0 \Rightarrow \tau = \tau_0, \sigma = \sigma_0, \tilde{\tau} = 0, \tilde{\sigma} = 0$$

The stress field is uniform within the layer

Stability analysis of undrained deformation

Fluid mass balance equations

$$\kappa \frac{\partial^2 \tilde{p}}{\partial z^2} = \frac{\partial \tilde{\varepsilon}}{\partial t}$$

$$\frac{\partial \tilde{p}}{\partial t} = \kappa M \frac{H}{H + \mu\beta M} \frac{\partial^2 \tilde{p}}{\partial z^2}$$

Constitutive equations

$$\tilde{\gamma} = \frac{\mu}{H} \tilde{p}$$

$$\tilde{\varepsilon} = -\frac{1}{M} \tilde{p} + \frac{\beta\mu}{H} \tilde{p}$$

The spatial dependence of the perturbations is decomposed into Fourier modes with wavelenth λ:

$$\tilde{p} = P(0) e^{st} \cos\left(\frac{2\pi z}{\lambda}\right)$$

$\lambda = L/n$, n is an integer, the zero fluid flux boundary conditions at $y = \pm L/2$ is satisfied.

s is the growth coefficient in time of the perturbation

$$s = -\kappa M \frac{H}{H + \mu\beta M} \left(\frac{2\pi}{\lambda}\right)^2$$

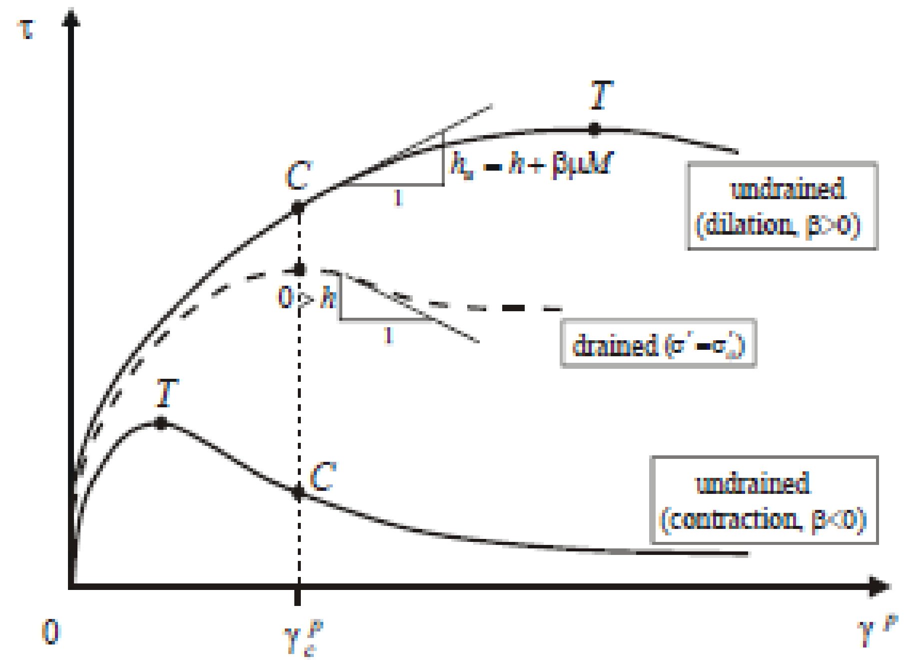
Stability analysis of undrained deformation

$$\text{if } \frac{H}{H + \mu\beta M} < 0, \text{ then } s > 0$$

$$\frac{\partial \tilde{p}}{\partial t} = c\frac{\partial^2 \tilde{p}}{\partial z^2} \text{ with } c < 0$$

This is equivalent to a diffusion equation with negative diffusivity or a diffusion equation with time running backwards: Non-uniformities become more localized rather than more diffuse with time.

- ✓ For dilatant material, instability occurs when H passes through zero from positive to negative: i.e. For dilatant hardening materials instability occurs when the underlying drained response passes through a peak.
- ✓ For compacting materials, the denominator changes sign when the undrained modulus h_u passes through zero



Stability analysis of undrained deformation

$$s = -\kappa M \frac{H}{H + \mu\beta M}\left(\frac{2\pi}{\lambda}\right)^2$$

- ✓ For dilatant materials, instability occurs for $H < 0$.

  The shortest wave length grows most rapidly.

  The perturbation wavelength cannot be arbitrarily small as it is physically bounded by the material microfabric length (grain size).

- ✓ For compacting materials, at the peak of the undrained stress-strain curve, $h_u=0$, and the perturbation rate is infinite for all wave lengths (ill-posedness).

- ✓ Necessity to incorporate more physics to regularize the mathematical ill-posedness:

  Constitutive model with microstructure

  Rate sensitivity

  Inertia

Stability analysis of undrained shearing for rate-sensitive materials

(Vardoulakis, 1996, Garagash, 2005)

Yield condition:

$$\tau = f\left(\sigma - p, \gamma^p, \dot{\gamma}^p\right)$$

$$\mu = \frac{\partial f}{\partial(\sigma - p)}, h = \frac{\partial f}{\partial \gamma^p}, \delta = \frac{\partial f}{\partial \ln \dot{\gamma}^p} \text{ (rate and state friction law)}$$

Governing equations for the perturbation quantities

$$\tilde{\tau} = -\mu \tilde{p} + h\tilde{\gamma}^p + \delta \frac{\tilde{\dot{\gamma}}^p}{\dot{\gamma}_0^p} = 0; \quad \tilde{\varepsilon} = \tilde{\varepsilon}^e + \tilde{\varepsilon}^p = \frac{\tilde{p}}{M} + \beta \tilde{\gamma}^p$$

$$\frac{\partial \tilde{\varepsilon}}{\partial t} = \kappa \frac{\partial^2 p}{\partial z^2}$$

$$\boxed{\frac{\partial}{\partial t}\left(h\tilde{\gamma}^p + \delta \frac{\tilde{\dot{\gamma}}^p}{\dot{\gamma}_0^p} + M\beta\mu\tilde{\gamma}^p\right) = \kappa M \frac{\partial^2}{\partial z^2}\left(h\tilde{\gamma}^p + \delta \frac{\tilde{\dot{\gamma}}^p}{\dot{\gamma}_0^p}\right)}$$

Stability analysis of undrained shearing for rate-sensitive materials

$$\tilde{\gamma}^{p} = \Gamma e^{st} \cos\left(\frac{2\pi z}{\lambda}\right)$$

Quadratic equation for the growth coefficient s

$$\frac{\delta}{\dot{\gamma}_0^p} s^2 + \left(h + M\beta\mu + M\kappa \frac{\delta}{\dot{\gamma}_0^p} \left(\frac{2\pi}{\lambda}\right)^2 \right) s + M\kappa h \left(\frac{2\pi}{\lambda}\right)^2 = 0$$

If a solution has a positive real part, then the corresponding perturbation grows exponentially in time.

Condition for the existence of positive growth coefficient

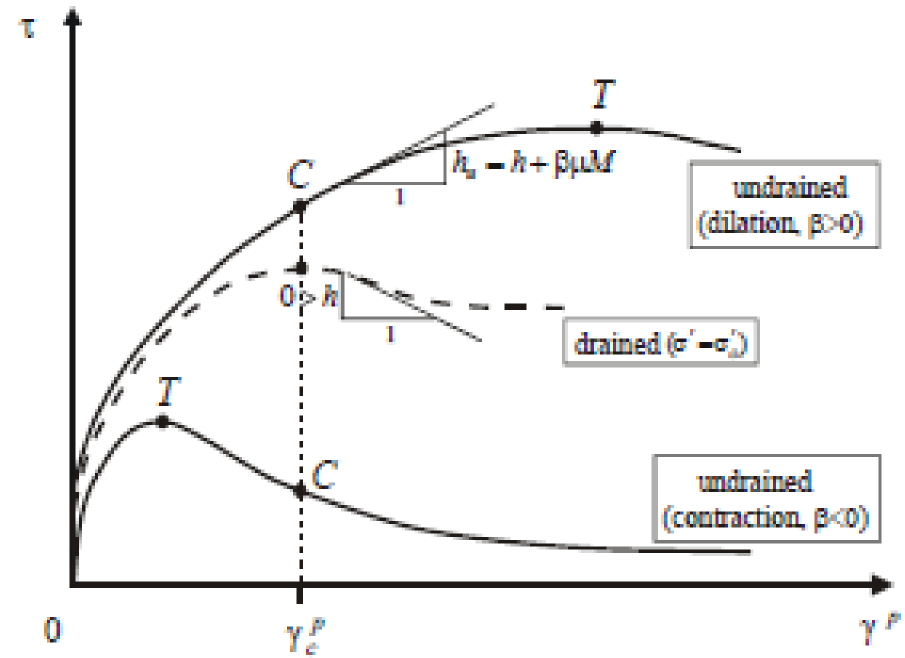
$$h + M\beta\mu + M\kappa \frac{\delta}{\dot{\gamma}_0^p} \left(\frac{2\pi}{\lambda}\right)^2 < 0$$

This condition occurs first for the largest wave length i.e. $\lambda = L$ i.e. diffuse instability occurs first for undrained shearing of compacting materials

Stability analysis of undrained shearing for rate-sensitive materials

Stability condition

$$h_u = h + M\beta\mu > -\frac{\kappa\delta}{\dot{\gamma}_0^p}\left(\frac{2\pi}{L}\right)^2$$



- ✓ Strain-rate sensitivity is amplified by pore fluid diffusion to delay the instability.
- ✓ Higher shearing rate leads to earlier instability.
- ✓ If the layer thickness is small enough, diffuse instability may be prevented (scale effect) and localized instability will occur first.

CONCLUSIONS

- ✓ Strain localization occurs when there exists one direction for which *acoustic tensor becomes singular* (Rice's criterion)
- ✓ Advanced constitutive models (non associate, non coaxial) allow for good predictions of the occurrence of strain localization and of the orientation of the shear band
- ✓ Classical continuum models are *unable to describe the finite thickness* of the shear band
- ✓ The underlying mathematical problem becomes *ill-posed* in the post localization regime
- ✓ Necessity to resort to continuum models with internal lengths (*micromorphic continuum models*) to regularize the mathematical problem in the post localization regime and to restore ellipticity for FEM applications.
- ✓ For saturated porous media, instability is controlled either by the *drained or the undrained properties*, depending on the constitutive equations and on the loading path. The inelastic volume changes play a key role in the occurrence of localized or diffuse instability.