26th ALERT Doctoral School, Modelling of instabilities and bifurcation in Geomechanics, Aussois, France 2016/10/06-08

Numerical modeling of bifurcation: Applications to borehole stability, multilayer buckling and rock bursting

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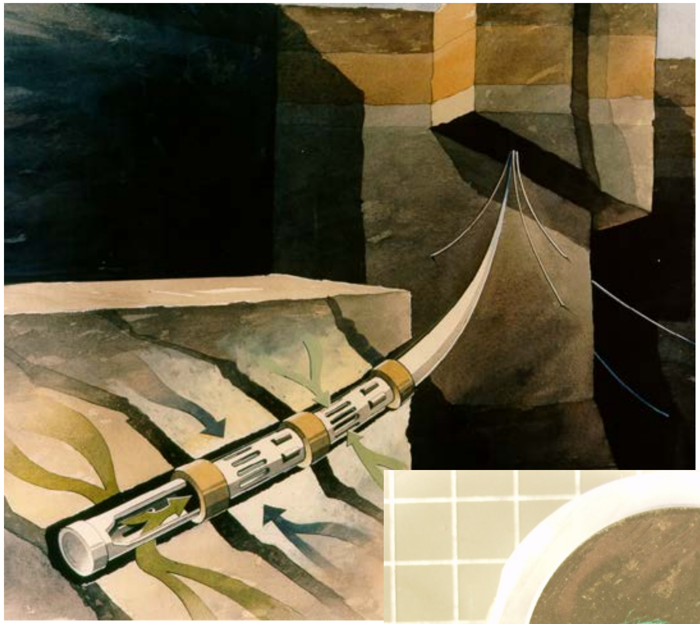
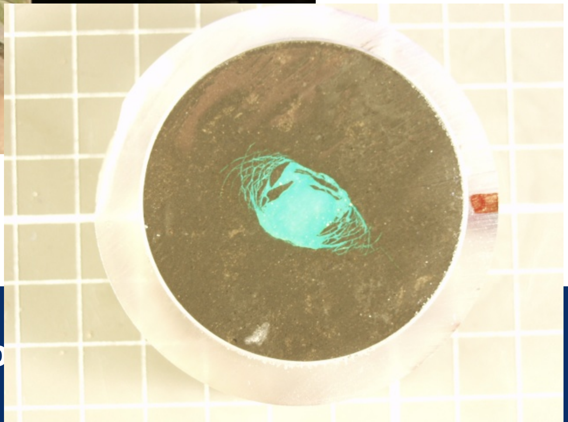
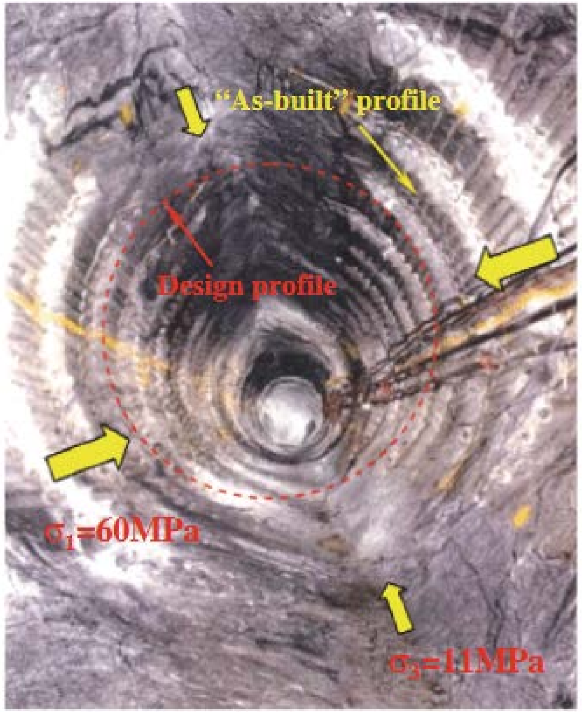
Typical boundary value bifurcation problems in geomechanics

1. Borehole stability with applications in **Oil and gas industry** and **Tunneling**







Martin 1997

2. Multi-Layer buckling with application in the folding of geological formations

3. Spalling and buckling of surface parallel cracks with application in rock bursting in mining

He et al. 2012

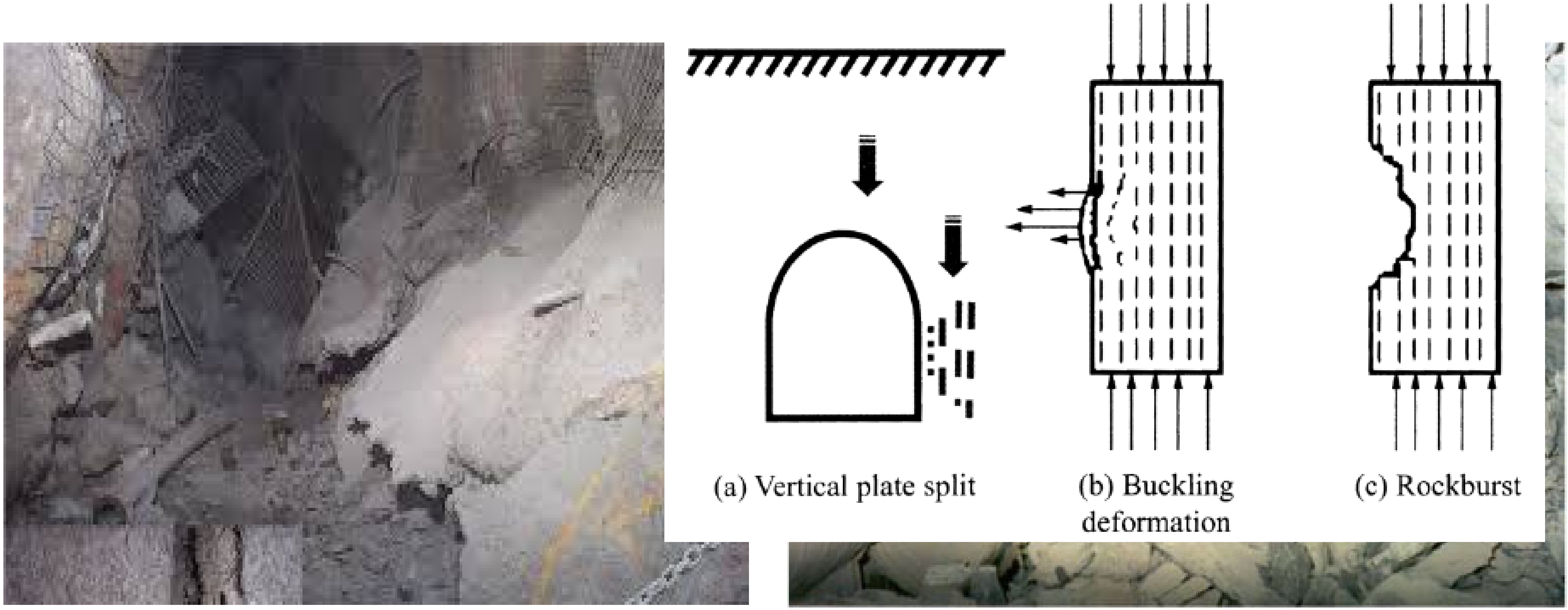
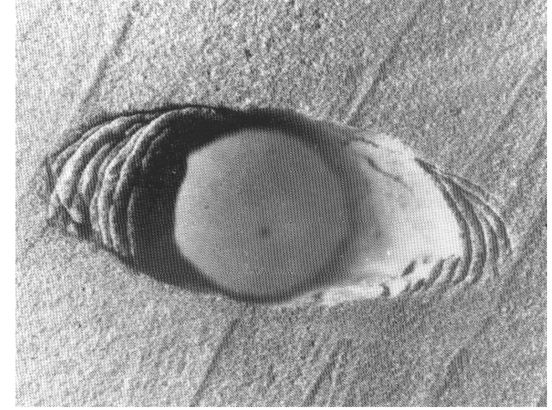
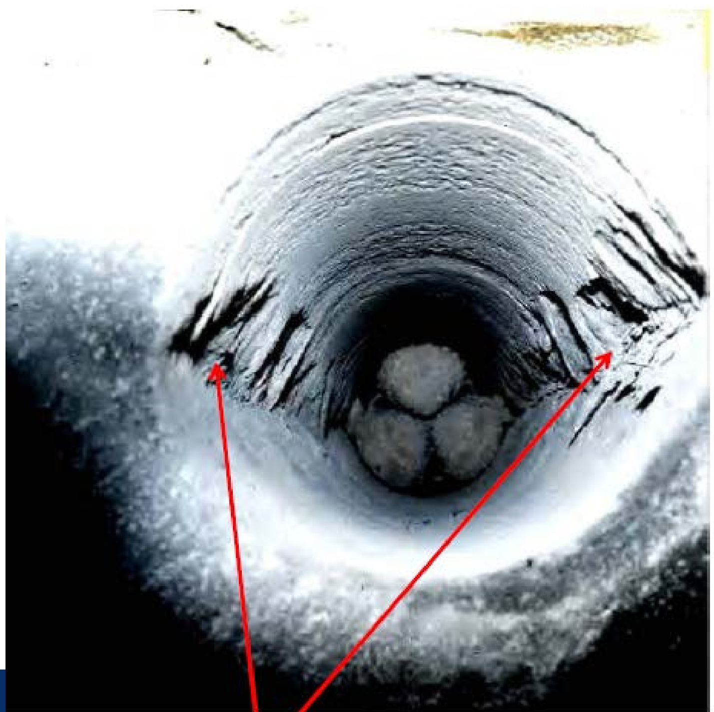


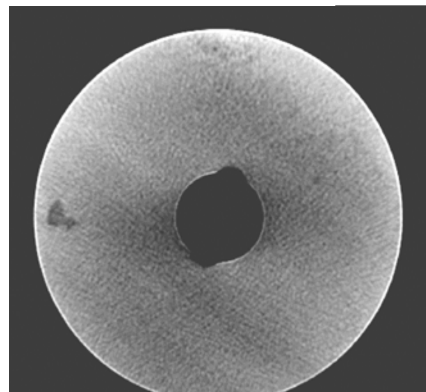
Figure 1—Rockburst damage and fragmentation (photograph W.D. Ortlepp)

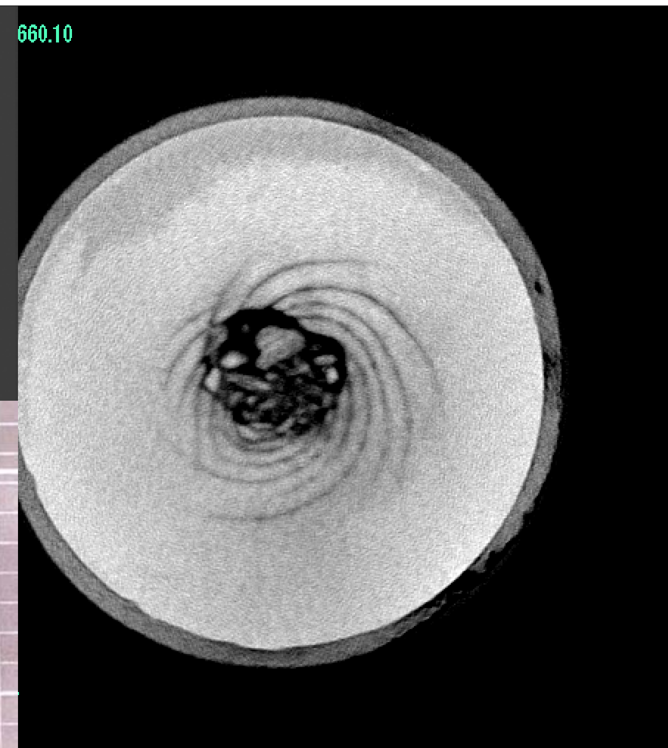
1 Borehole stability



- Borehole failure is an important geomechanical problem for the assessment of the integrity of tunnels, wellbores and perforations in the field
- A common failure mode is the formation of breakouts







Breakout failure modes (Maury 1992)

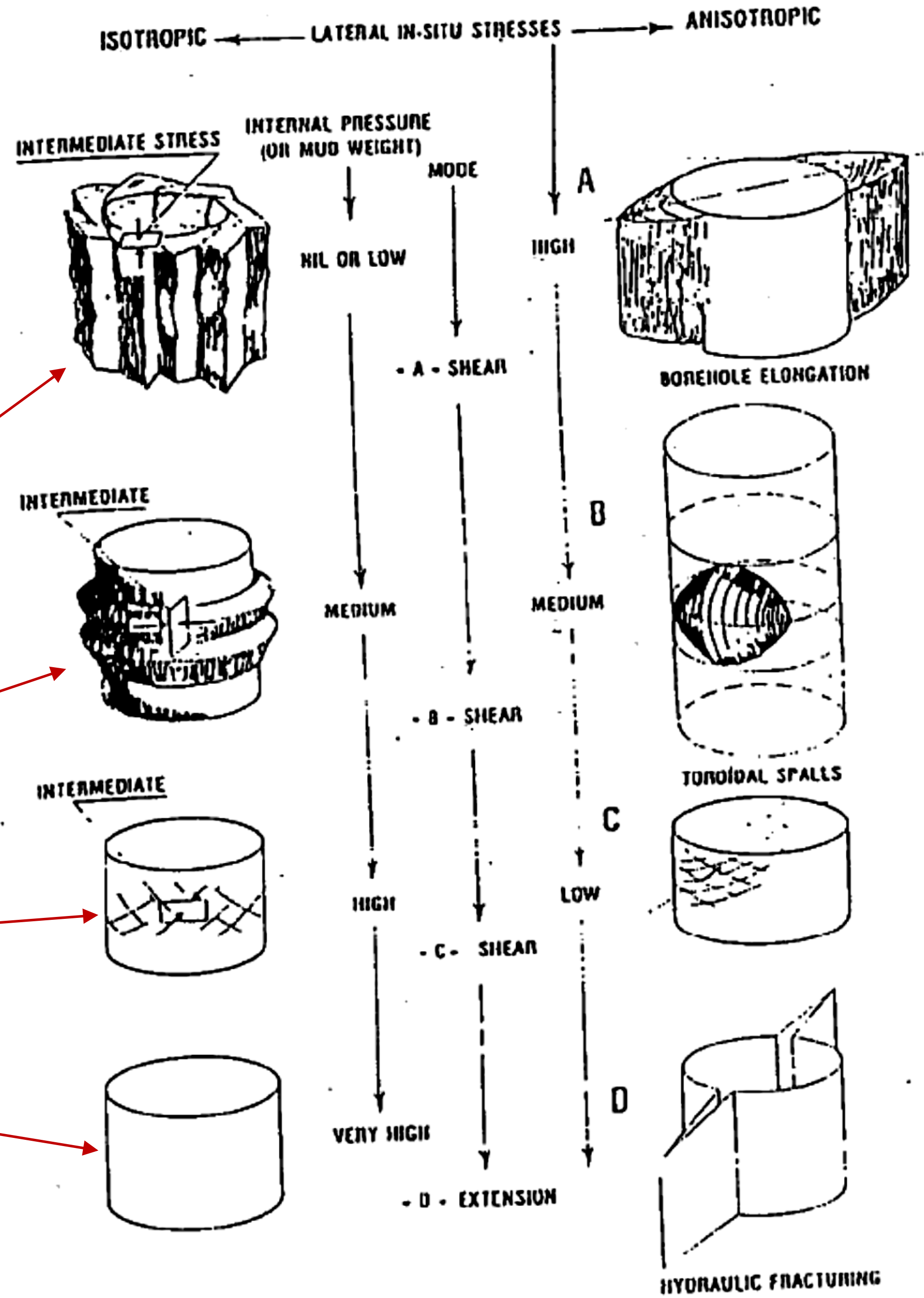
- Orientation of breakout failure depends on the relative stress magnitude of

$$\sigma_r \le \sigma_z \le \sigma_\theta$$

$$\sigma_r \le \sigma_\theta \le \sigma_z$$

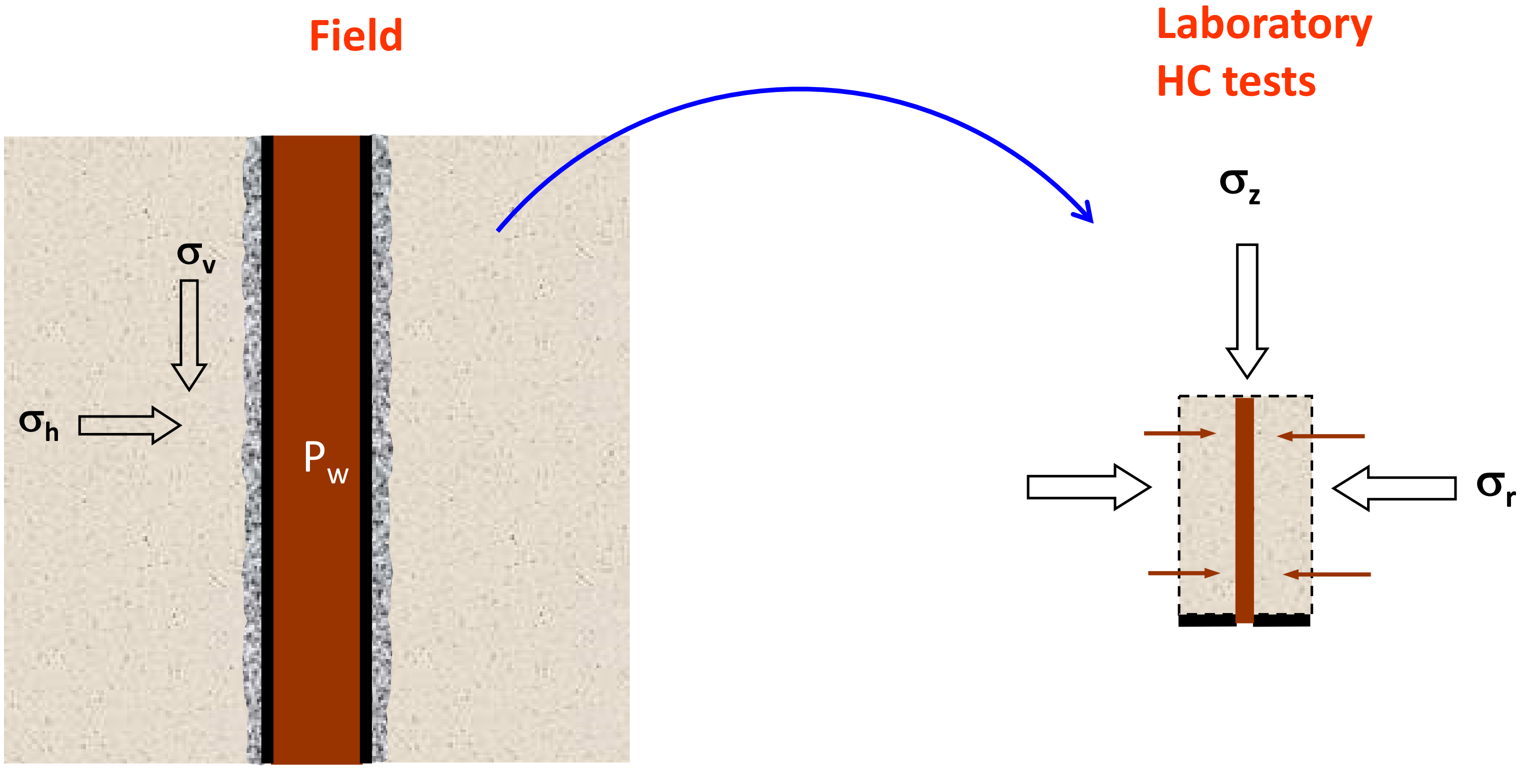
$$\sigma_z \le \sigma_r \le \sigma_\theta$$

$$\sigma_\theta \le \sigma_z \le \sigma_r$$



Hollow cylinder test

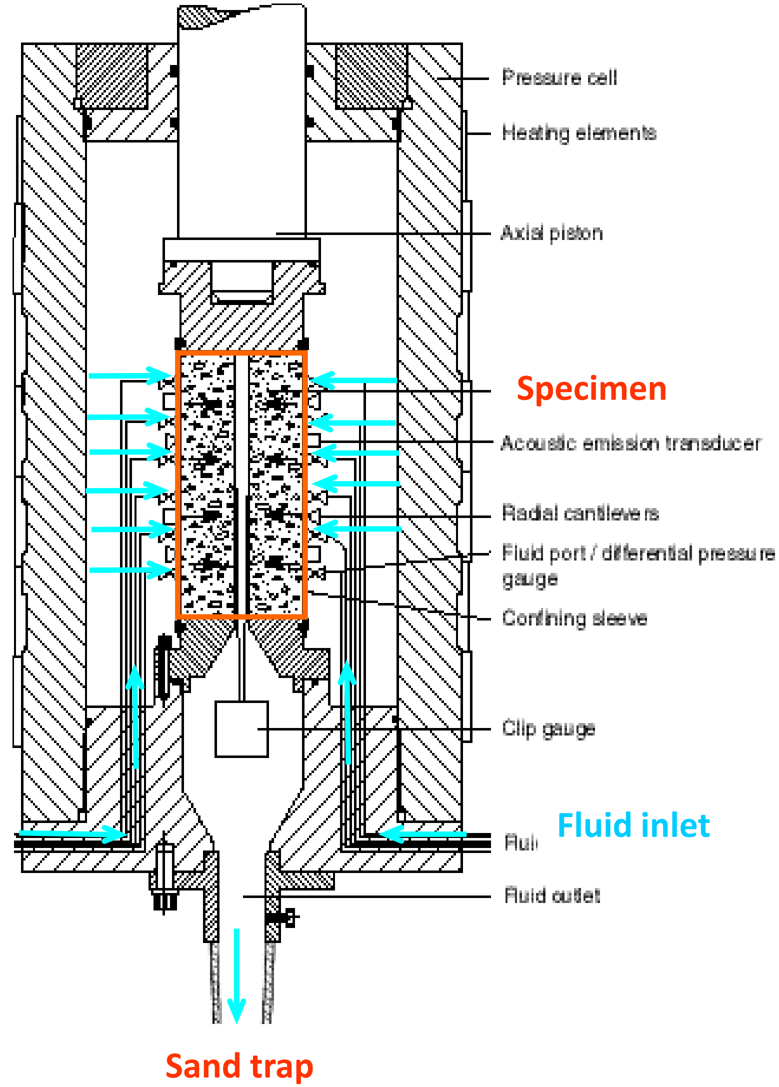
- Typical test for studying hole failure for borehole stability



Hollow cylinder test (with fluid flow)

(Papamichos et al. 2001)

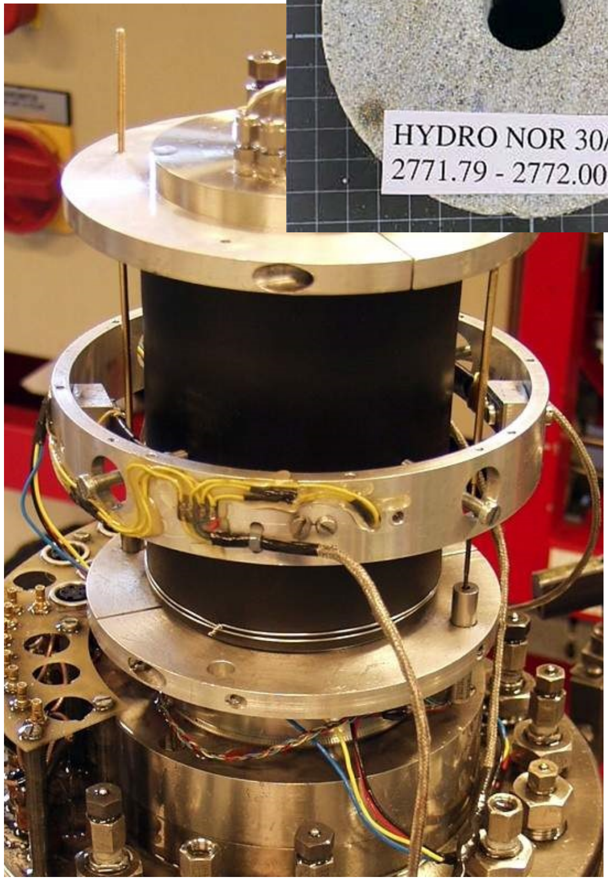
- Typical system
  (SINTEF Petroleum Research, Norway)
  - Specimen size:
    o.d. 10/20 cm, i.d. 2/5 cm
  - Isotropic confining stress up to 100 MPa
  - Oil / Water flow up to 4 L/min or 40 MPa fluid pressure
  - Gas flow
  - Temperature up to 80°C
  - Axial, internal and external diametrical deformations
  - Continuous sand production measurements
  - Radial permeability measurements



Hollow cylinder experiments

HC sample from a North Sea core

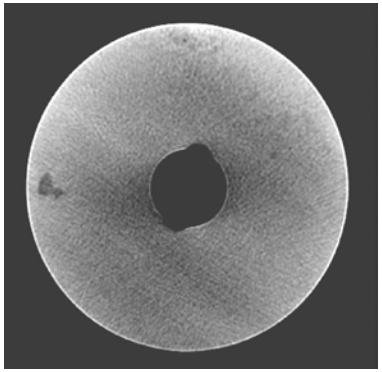
Loading cell

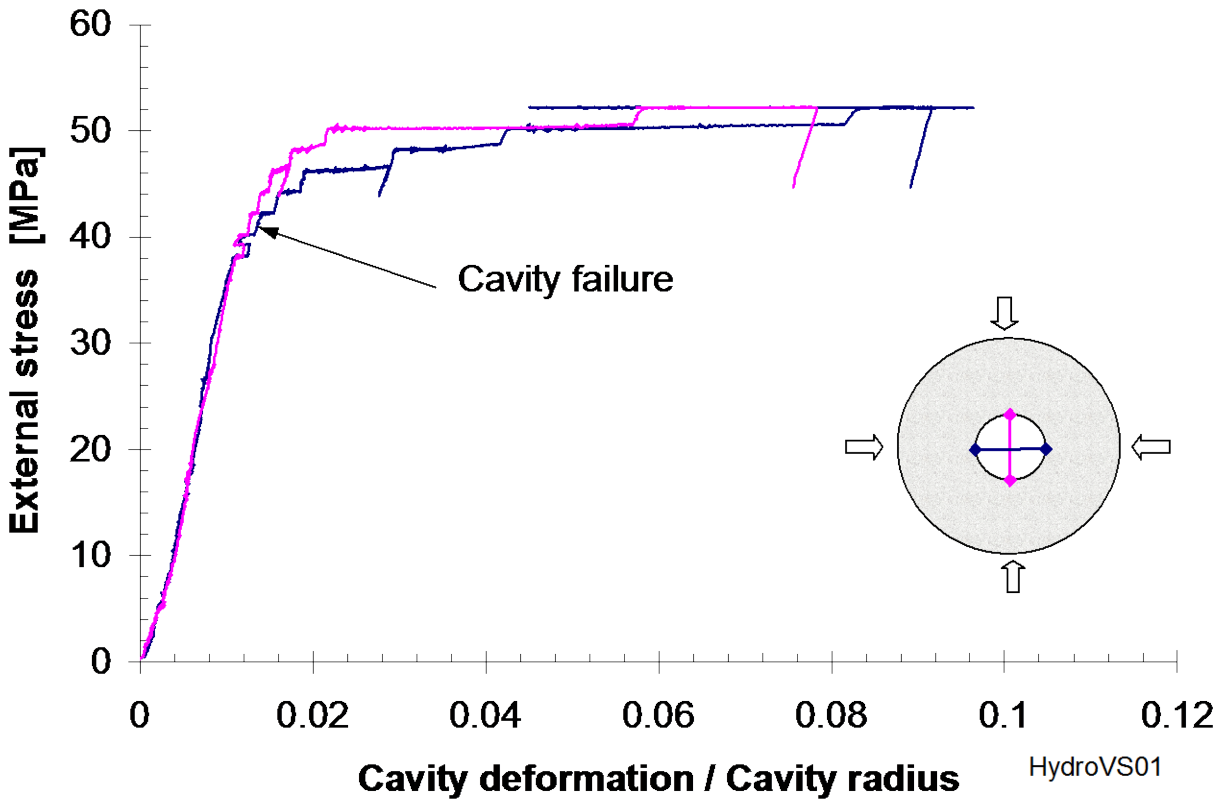


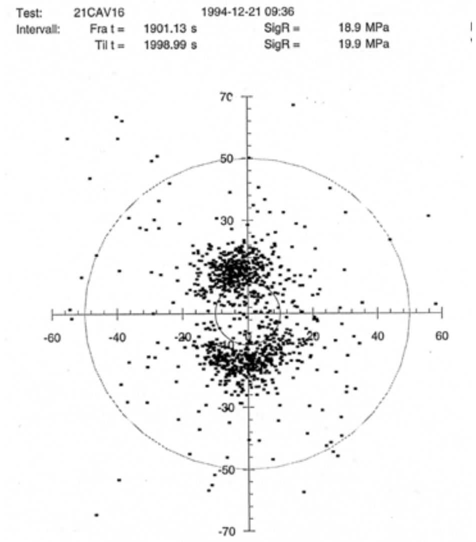
Instrumented jacketed specimen

Photographs SINTEF Petroleum Research, Norway

- Cavity deformations
  - The deviation of the 2 measurements indicates cavity failure
  - AE location and borescope data confirm this



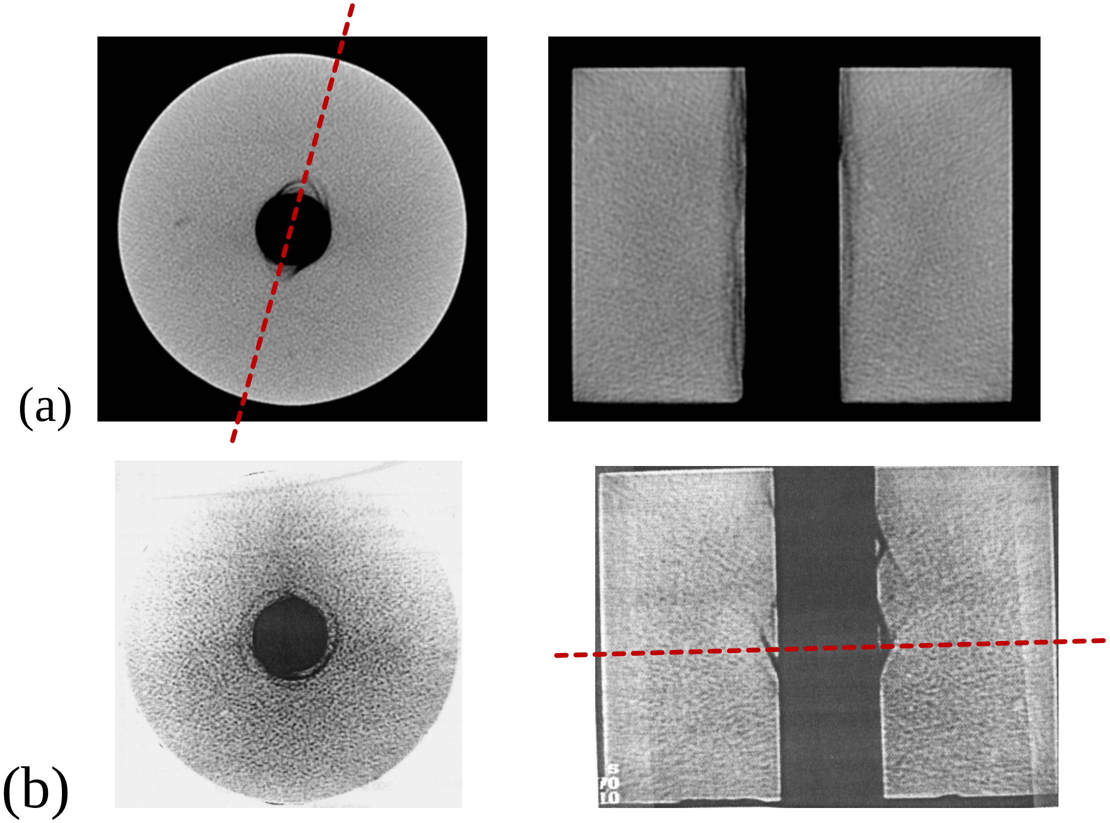




Hollow cylinder failures

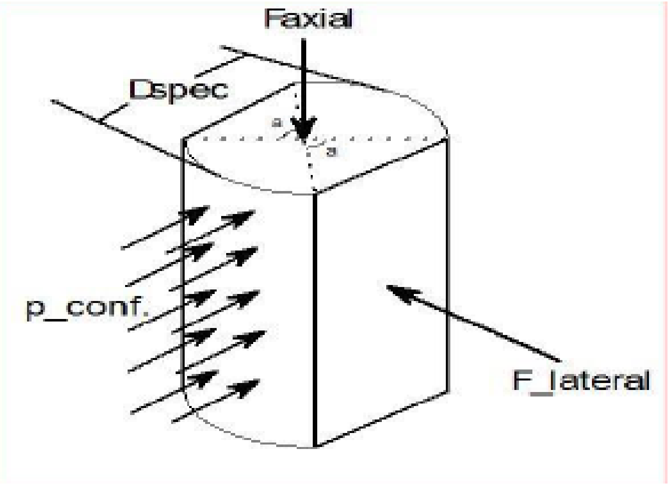
Effect of stress anisotropy on failure pattern

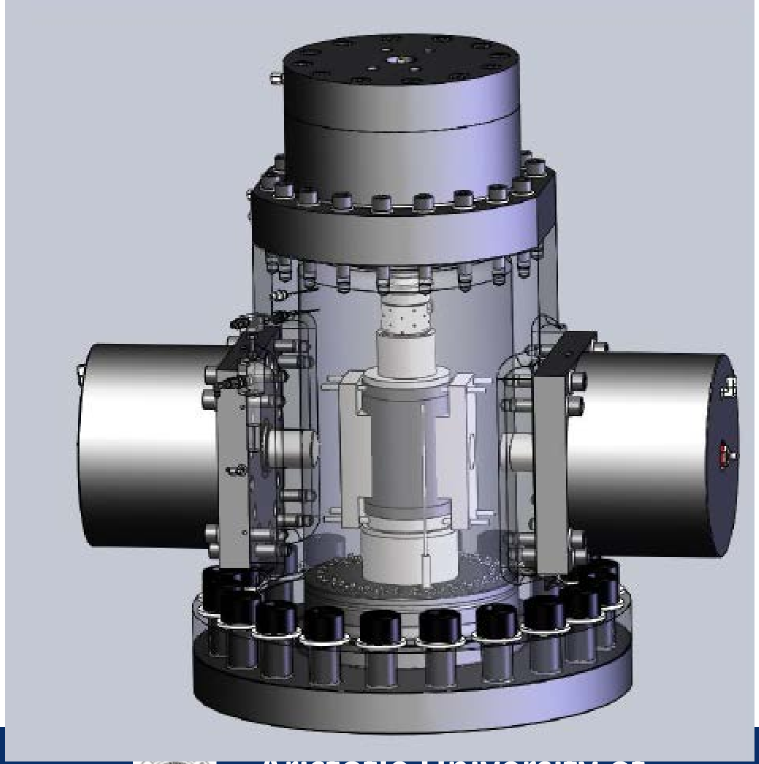
- CT scan sections normal and parallel to the hole axis
- Cavity failure on Red Wildmoor sandstone:
  - (a) high lateral stress
  - (b) high axial stress



True triaxial apparatus

- MTS True triaxial cell
  - 100 MPa confining stress
  - 50 MPa vertical deviatoric stress
  - 50 MPa horizontal deviatoric stress
  - Truncated samples, 200 mm diameter
  - Hollow-cylinder tests
  - Radial flow



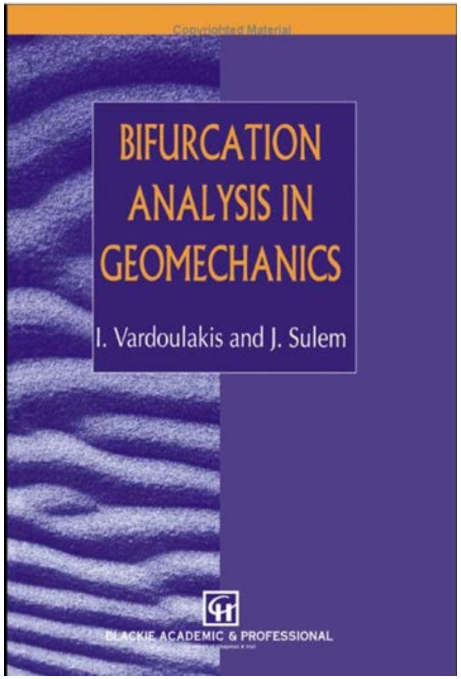


Why bifurcation theory ?

- Underlying hypothesis of the bifurcation theory is that breakouts and shear bands are the result of material instabilities, termed equilibrium bifurcations
- Bifurcation approach associates failure with the occurrence of the instabilities in contrast to classical procedures where failure is usually assumed ad hoc to be an intrinsic material property associated with the elastic-plastic limit
- Elastoplastic solutions associated with a yield/failure surface UNDER-PREDICT hole failure
  - Conservative
- NO SIZE effect for the hole strength is predicted
  - Small holes stronger than large holes

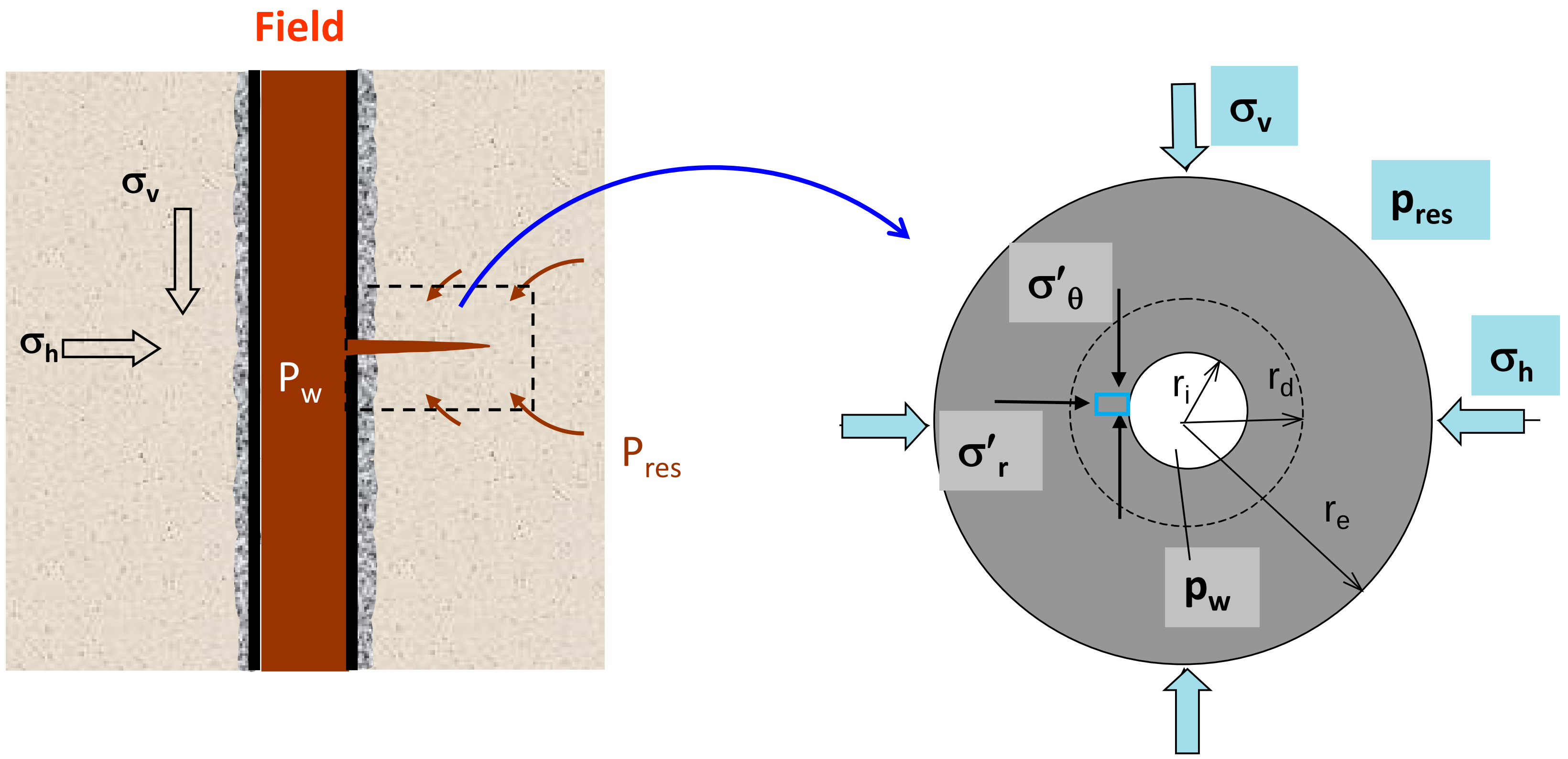
- Many studies from the late 1980's studied hole failure as a bifurcation phenomenon
  - Vardoulakis I, Papanastasiou P, Papamichos E, Sulem J, Guenot A, van den Hoek PJ, etc.

- From Foreword by Cor J Kenter, Head Rock Mechanics, Shell Research
  - "Suddenly a technology that was initially regarded as rather academic contributes to million dollars savings."



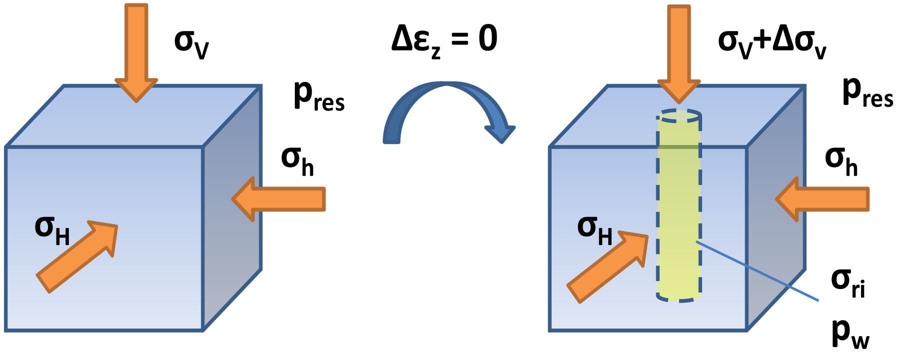
Physical boundary value problem

- In situ stresses $\Rightarrow$ Stress concentrations around the hole
  - Elasticity: Kirsch solution
  - Elasto-plasticity: Solutions for Mohr Coulomb *F*, *Q*



Drilling of borehole/tunnel

- Stresses around a borehole essential for any borehole related problem
  - Drilling, production, hydraulic fracturing, water injection, waste disposal, CO_2 injection, tunneling, mining, etc.



Stresses in cylindrical coordinates

- 3 normal stresses: Radial σ_r, tangential σ_θ and axial σ_z
- 3 shear stresses: $\sigma_{r\theta}$ (or $\tau_{r\theta}$), $\sigma_{\theta z}$ (or $\tau_{\theta z}$), σ_{rz} (or τ_{rz})

$$\sigma_r = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta$$

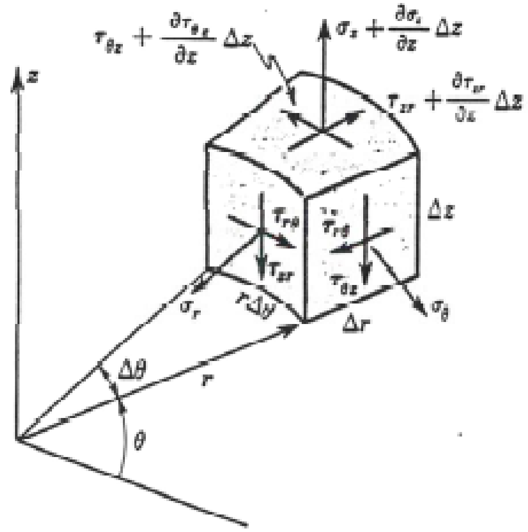
$$\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\sigma_z = \sigma_z$$

$$\tau_{r\theta} = \frac{1}{2}(\sigma_y - \sigma_x)\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\tau_{rz} = \tau_{xz}\cos\theta + \tau_{yz}\sin\theta$$

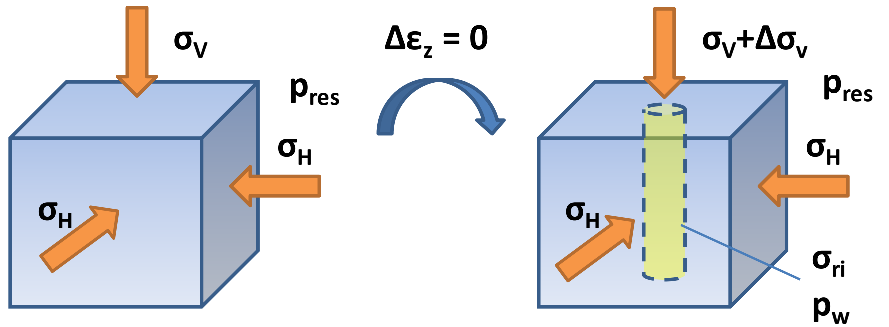
$$\tau_{\theta z} = \tau_{yz}\cos\theta - \tau_{xz}\sin\theta$$



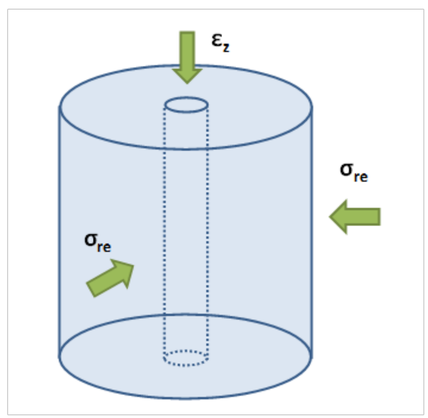
Axisymmetric problems

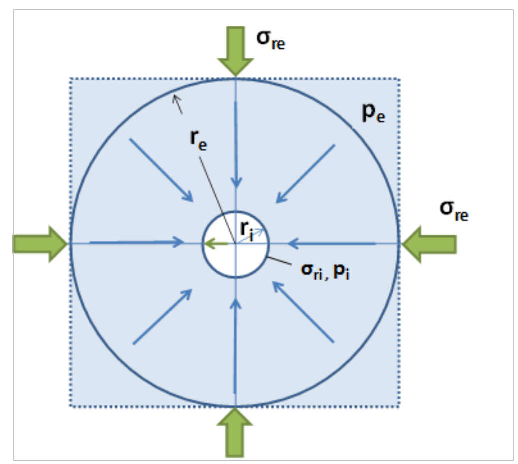
Examples

1. Drilling of a vertical borehole under isotropic in situ horizontal stresses



2. Hollow Cylinder (HC) test



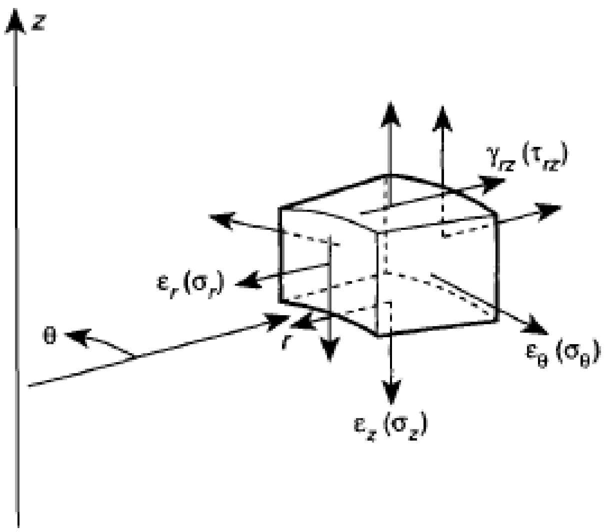


Simplifications due to axisymmetry

- $u_\theta = 0$
- $\sigma_{r\theta} = \sigma_{\theta z} = 0$
- $\varepsilon_{r\theta} = \varepsilon_{\theta z} = 0$

$$\varepsilon_r = \frac{\partial u_r}{\partial r}, \qquad \varepsilon_\theta = \frac{u_r}{r}$$

$$\varepsilon_z = \frac{\partial u_z}{\partial z}, \qquad \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$$



Strains and stresses involved in the analysis of axisymmetric solids.

- Equilibrium equations

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + f_r = 0$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\sigma_{rz}}{r} + f_z = 0$$

further…

- Usually in our problems (drilling, tunneling, HC, HF, ….)
- No shear : $\sigma_{rz} = \varepsilon_{rz} = 0$
- Body forces due to gravity: e.g. in a vertical hole: $f_r = 0, f_z = -\rho g$
  - $\Rightarrow$ equilibrium equations decouple and can be solved independently

$$\frac{\partial\sigma_r}{\partial r} + \frac{\partial\cancel{\sigma_{rz}}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + \cancel{f_r} = 0 \qquad \Rightarrow \qquad \frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\frac{\partial\cancel{\sigma_{rz}}}{\partial r} + \frac{\partial\sigma_z}{\partial z} + \frac{\cancel{\sigma_{rz}}}{r} + f_z = 0 \qquad \qquad \frac{d\sigma_z}{dz} = \rho g$$

Elastic solution for axisymmetric problem

- Substitution of σ - ε and $\varepsilon - u_r$ equations in equilibrium equations

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \quad \Rightarrow \quad \frac{d^2 u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} = 0$$

- One unknown u_r (pore pressure neglected for simplicity)

- Equilibrium eqn in terms of displacement u_r

$$\boxed{\frac{d^2 u_r}{dr^2} + \frac{1}{r}\frac{du_r}{\partial r} - \frac{u_r}{r^2} = 0 \quad \Rightarrow \quad \frac{d}{dr}\left(\frac{du_r}{dr} + \frac{u_r}{r}\right) = \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r u_r\right)\right] = 0}$$

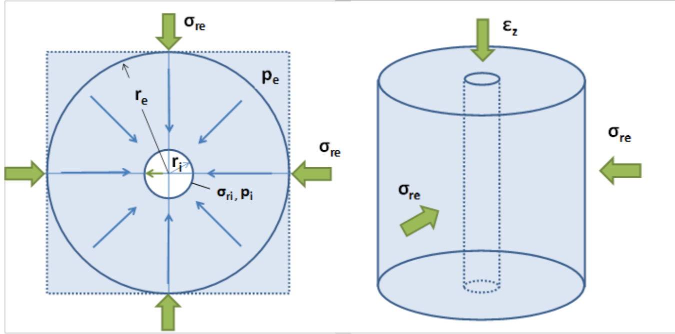
- Direct integration 2 times gives

$$u_r = c_1 r + \frac{c_2}{r}$$

- c_1 and c_2 are integration constants to be solved from the b.c.

$$\sigma_r\left(r = r_i\right) = \sigma_{ri} \,, \qquad \sigma_r\left(r = r_e\right) = \sigma_{re}$$

Elastic solution for axisymmetric problem



- Displacement u_r and strains ε_r and ε_θ

$$\varepsilon_r = \frac{1-2\nu}{2G}\sigma_{ri} + \frac{\sigma_{re}-\sigma_{ri}}{2G}\frac{1-2\nu - r_i^2/r^2}{1-r_i^2/r_e^2} - \nu\varepsilon_z$$

$$\varepsilon_\theta = \frac{u_r}{r} = \frac{1-2\nu}{2G}\sigma_{ri} + \frac{\sigma_{re}-\sigma_{ri}}{2G}\frac{1-2\nu + r_i^2/r^2}{1-r_i^2/r_e^2} - \nu\varepsilon_z$$

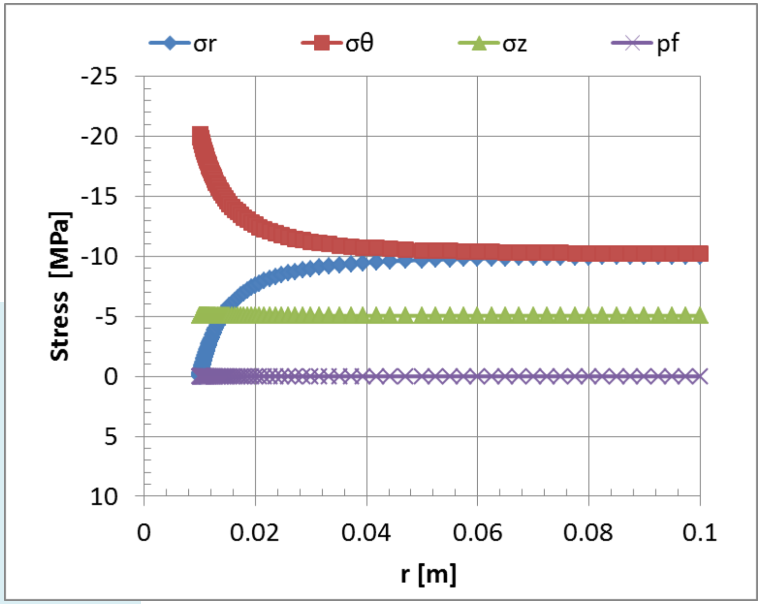
- Stresses σ_r, σ_θ, σ_z

$$\sigma_r = \sigma_{ri} + (\sigma_{re}-\sigma_{ri})\frac{1-r_i^2/r^2}{1-r_i^2/r_e^2}$$

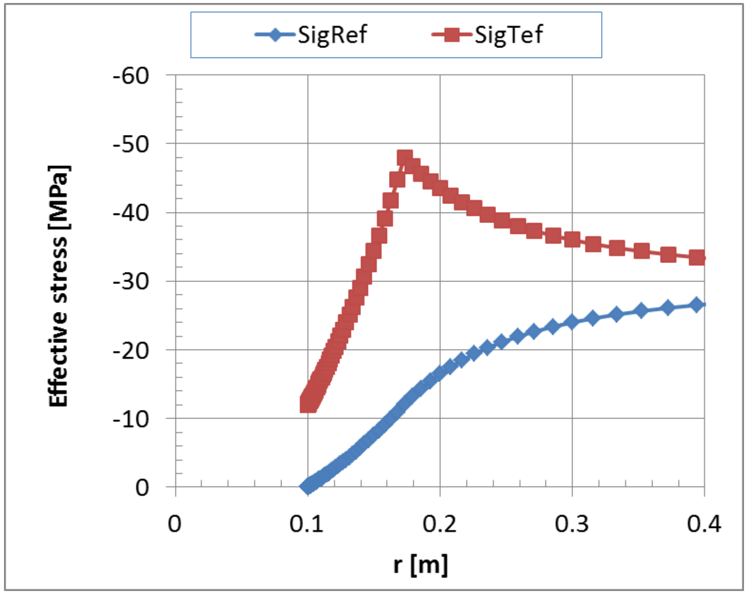
$$\sigma_\theta = \sigma_{ri} + (\sigma_{re}-\sigma_{ri})\frac{1+r_i^2/r^2}{1-r_i^2/r_e^2}$$

$$\sigma_z = \nu(\sigma_r+\sigma_\theta) + E\varepsilon_z = 2\nu\sigma_{ri} + (\sigma_{re}-\sigma_{ri})\frac{2\nu}{1-r_i^2/r_e^2} + E\varepsilon_z$$

Final stresses + pore pressure		
Internal radial stress [MPa]	SigRi =	0
External radial stress [MPa]	SigRe =	-10
Internal pore pressure [MPa]	pfi =	0
External pore pressure [MPa]	pfe =	0



Elastoplasticity - Plastic region



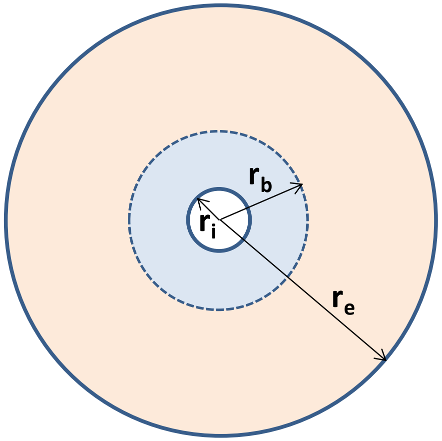
The stresses in the plastic region must satisfy:

- Equilibrium equation: $$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$
- Yield condition:

$$F = \frac{1}{2}(\sigma_r - \sigma_\theta) + \frac{1}{2}(\sigma_r + \sigma_\theta)\sin\varphi - c\cos\varphi = 0 \quad \Rightarrow$$

$$\Rightarrow \quad \sigma_\theta = (1+M)\sigma_r - MT_c$$

$$M = \frac{2\sin\varphi}{1-\sin\varphi}, \quad T_c = \frac{c}{\tan\varphi}$$



- Substitute in equilibrium equation: $$\frac{d\sigma_r}{dr} - M\frac{\sigma_r}{r} = -M\frac{T_c}{r}$$
- Solve equation with b.c. $$\sigma_r(r = r_i) = \sigma_{ri}$$

Elastoplastic boundary at $r = r_b$

- Solution is:

$$\sigma_r = (\sigma_{ri} - T_c)\frac{r^M}{r_i^M} + T_c$$

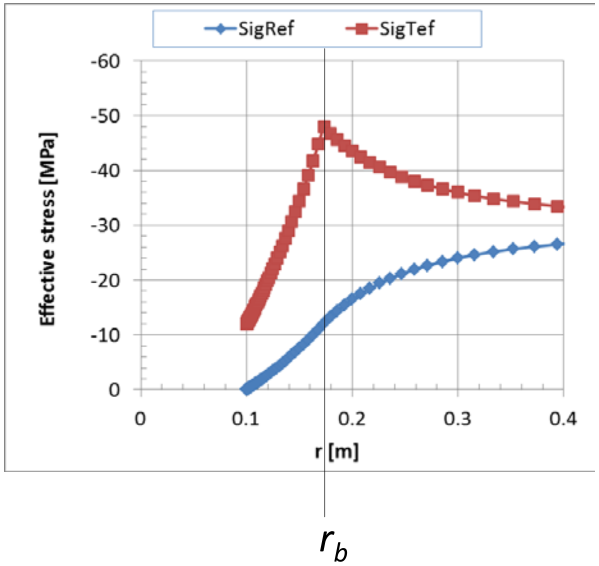
$$\sigma_\theta = (1+M)(\sigma_{ri} - T_c)\frac{r^M}{r_i^M} + T_c$$

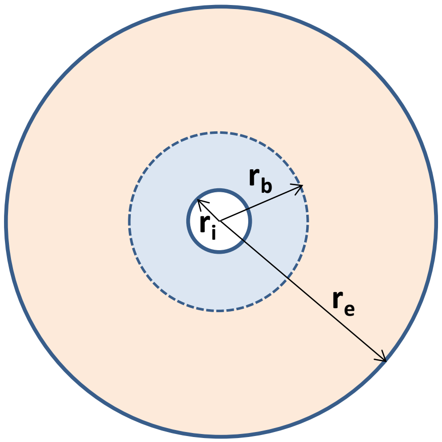
- At EP boundary:

$$\sigma_{rb} = (\sigma_{ri} - T_c)\frac{r_b^M}{r_i^M} + T_c$$

- Radial stress must be continuous with stress from elastic region

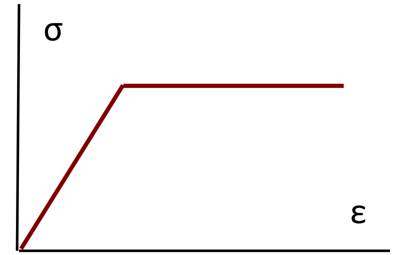
$$r_b = r_i\left[\frac{(T_c - \sigma_{re})(1-\sin\varphi)}{T_c - \sigma_{ri}}\right]^{1/M}$$



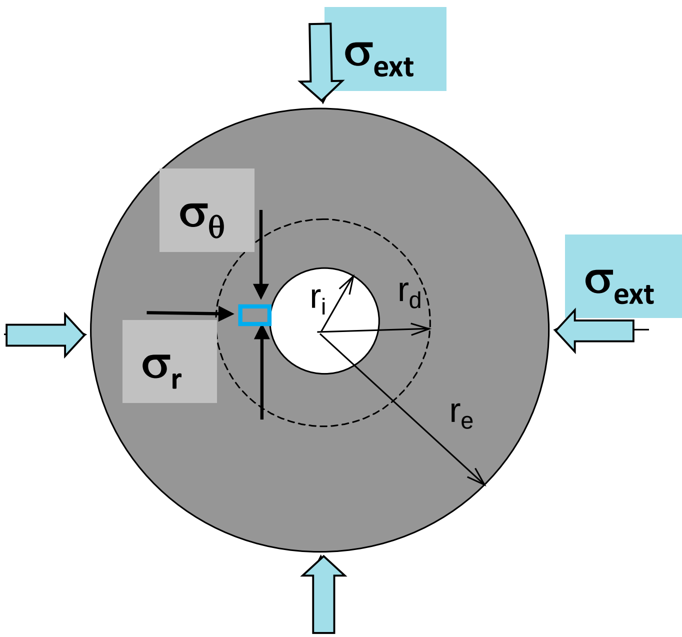


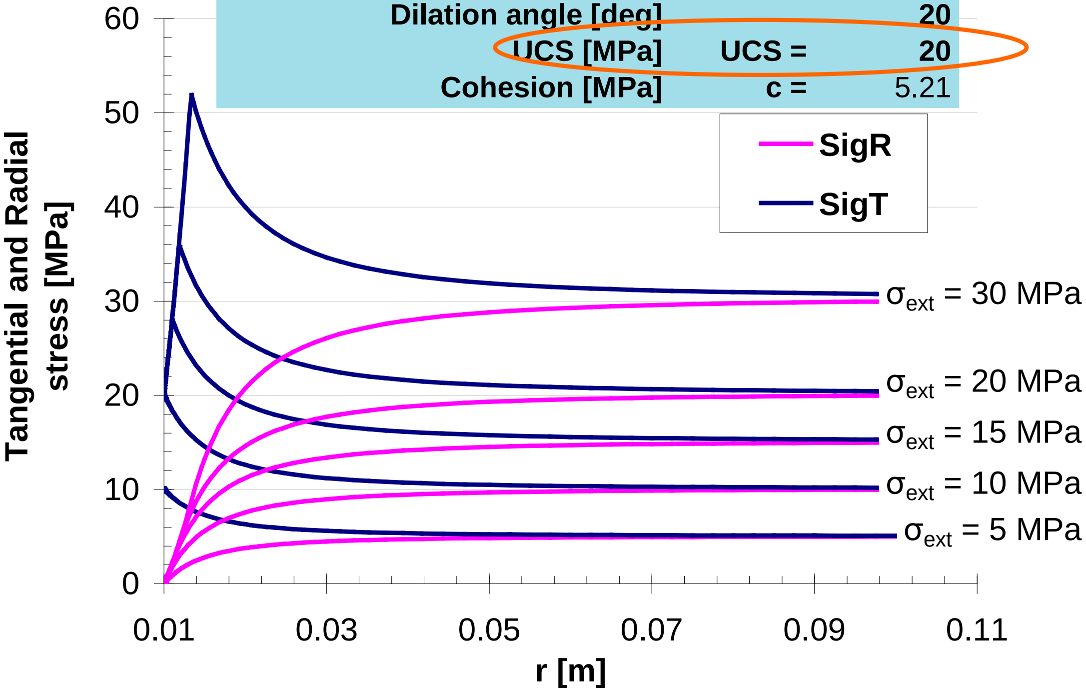
Stresses in the borehole

- Linear elasticity + perfect plasticity with Mohr-Coulomb yield function
- **Yield when σ_{ext} = ½ *UCS***
  (because of stress concentration)

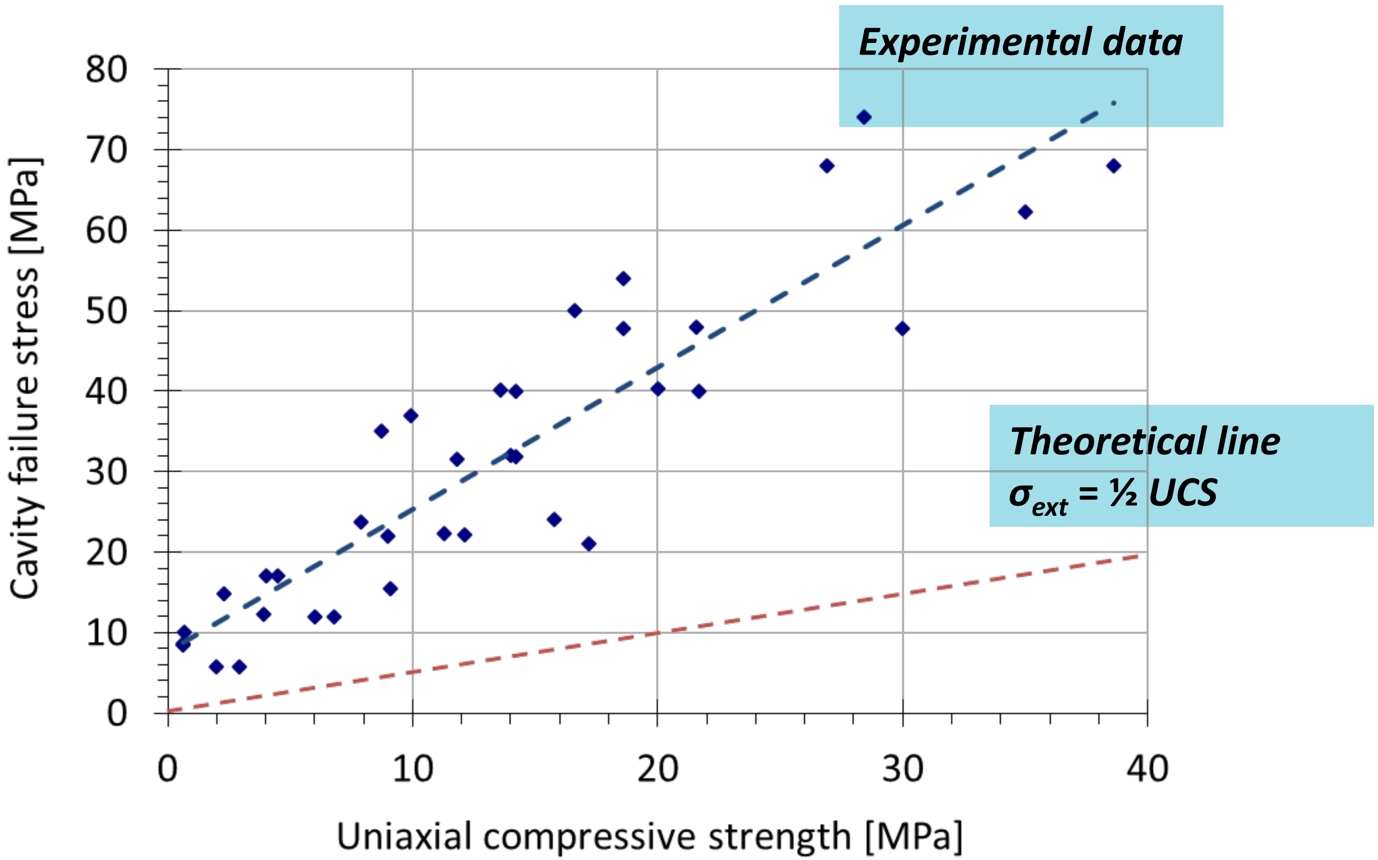


Material properties		
Elastic Young's modulus [MPa]	E =	5000
Poisson's ratio [-]	nu =	0.2
Biot's eff. Stress coeff. [-]	alfa =	1
Friction angle [deg]		35
Dilation angle [deg]		20
UCS [MPa]	UCS =	20
Cohesion [MPa]	c =	5.21



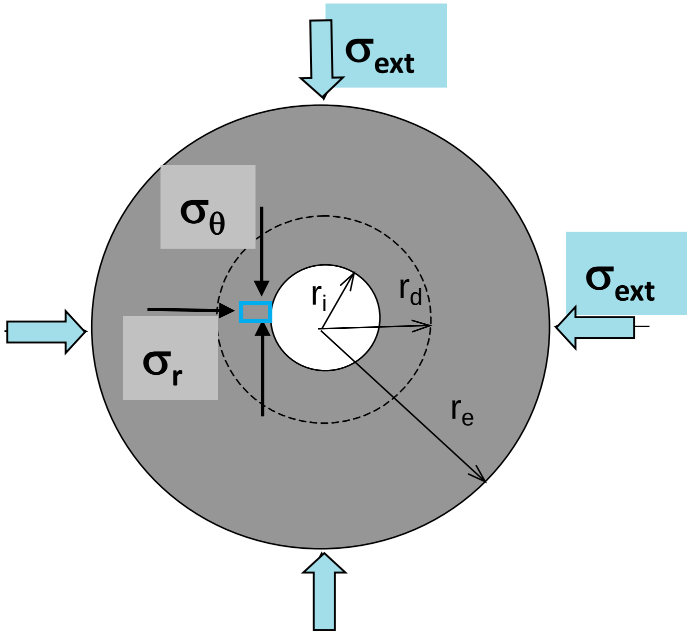


Cavity failure stress vs. UCS for sandstones

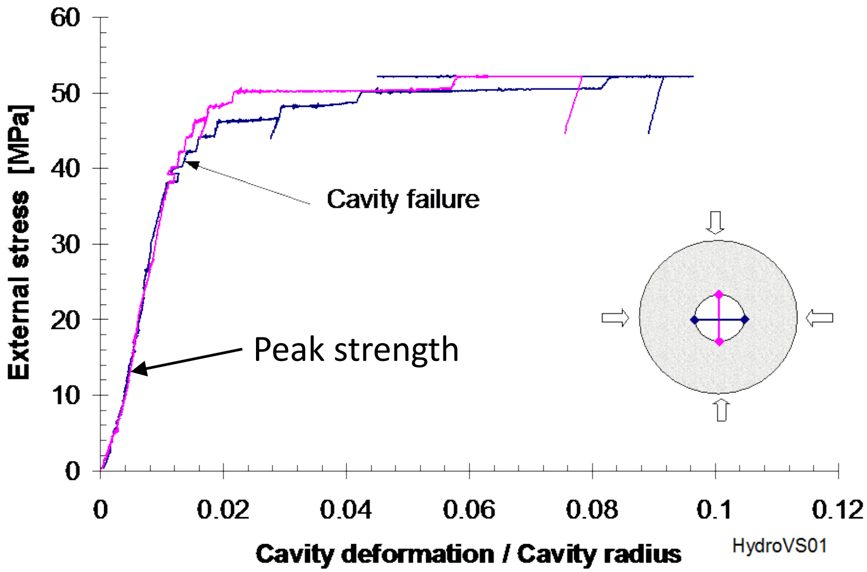


Why?

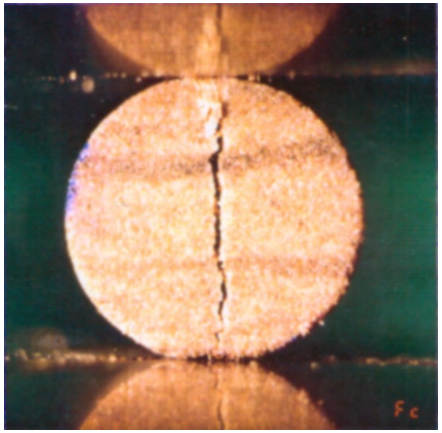
- Rock near the cavity does not fail when it reaches its peak strength

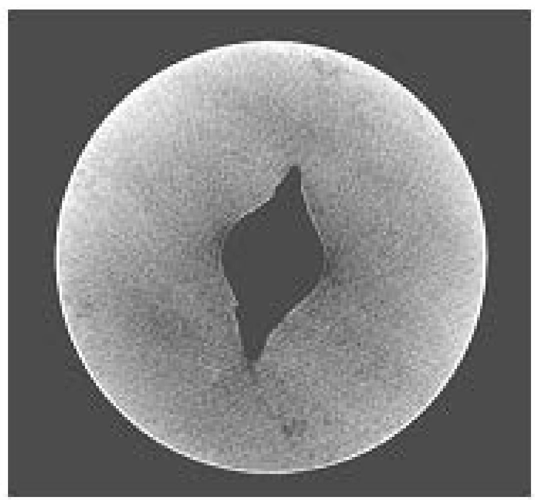


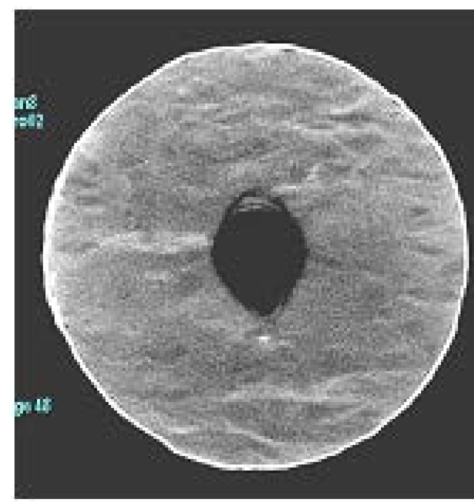
- Instead it yields and plastifies creating a plastic region
- Remaining rock supports more stress until macroscopic localization



Macroscopic rock failures in laboratory tests



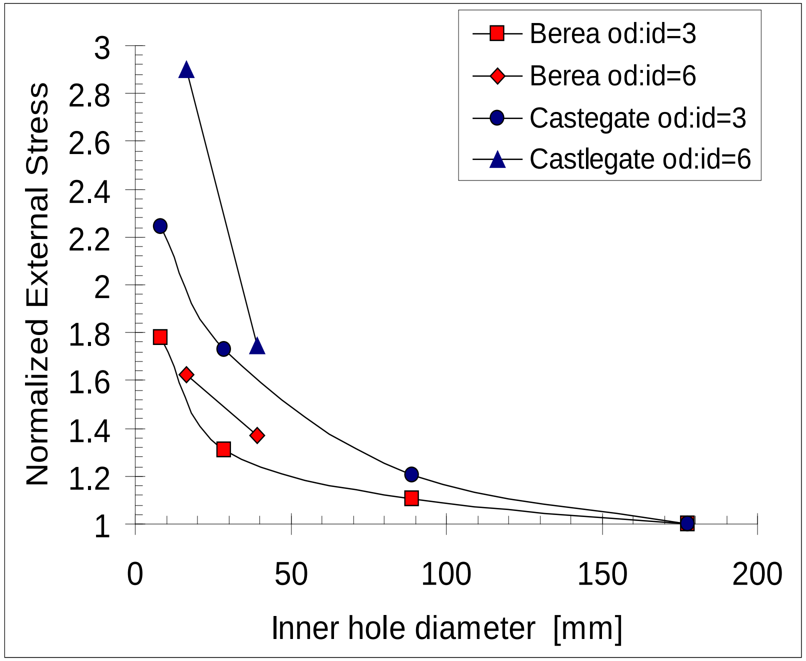




Size or scale effect on failure stress

- Any size hole in the earth (an infinite medium) has the same diameter
  - Classic theories
- Experiments show that:
  "Small holes are more stable than big holes"
  - Theories with internal length are needed

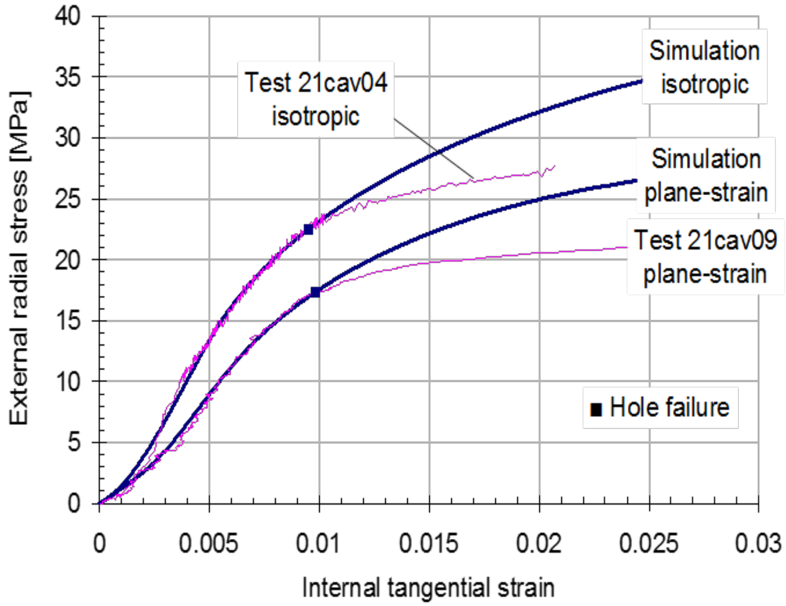
- NOTE: What is size/scale dependent is the localization failure and not the continuum behavior of the rock



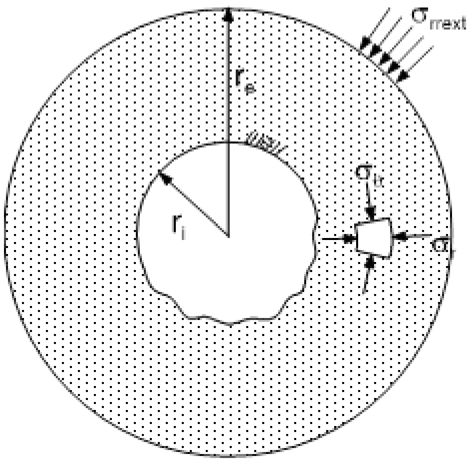
Formulation of bifurcation criterion within a FE scheme

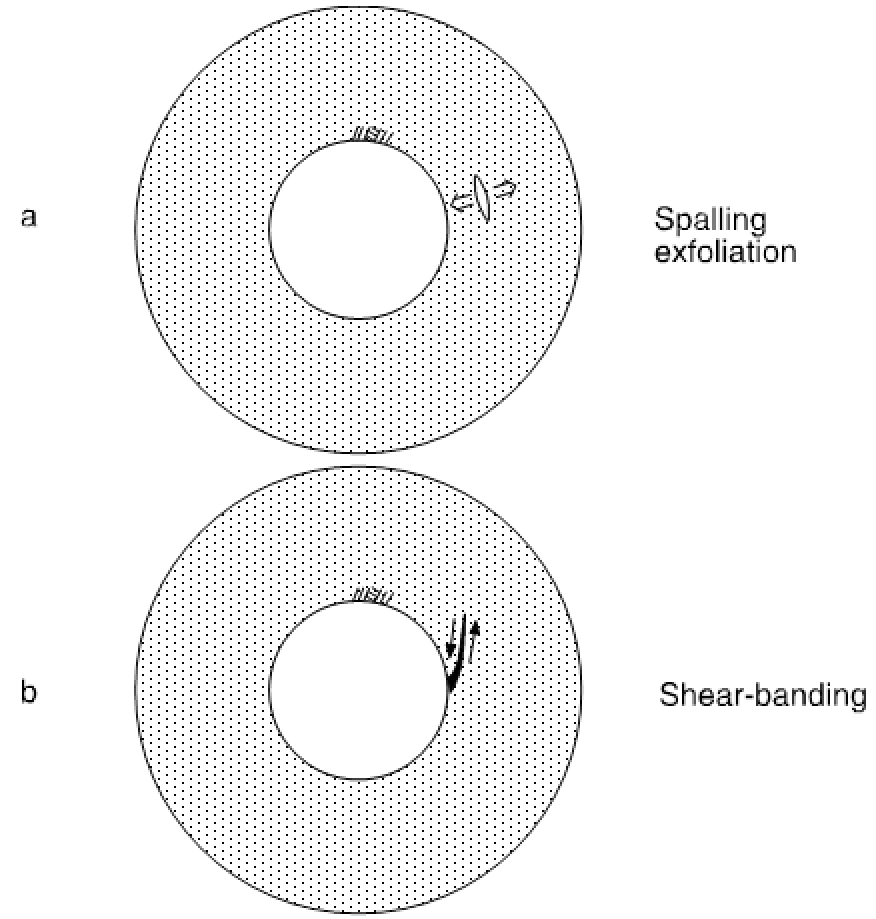
- Bifurcation criterion for the global BVP
- Criterion gives the onset of deviation/bifurcation of classical solution of uniform hole closure

- Formulation is based on the assumption that in addition to the trivial solution of cylindrical convergence of the hole during the primary loading path, there exists another non-trivial warping solution that fulfils homogeneous boundary conditions



Lateral borehole failure

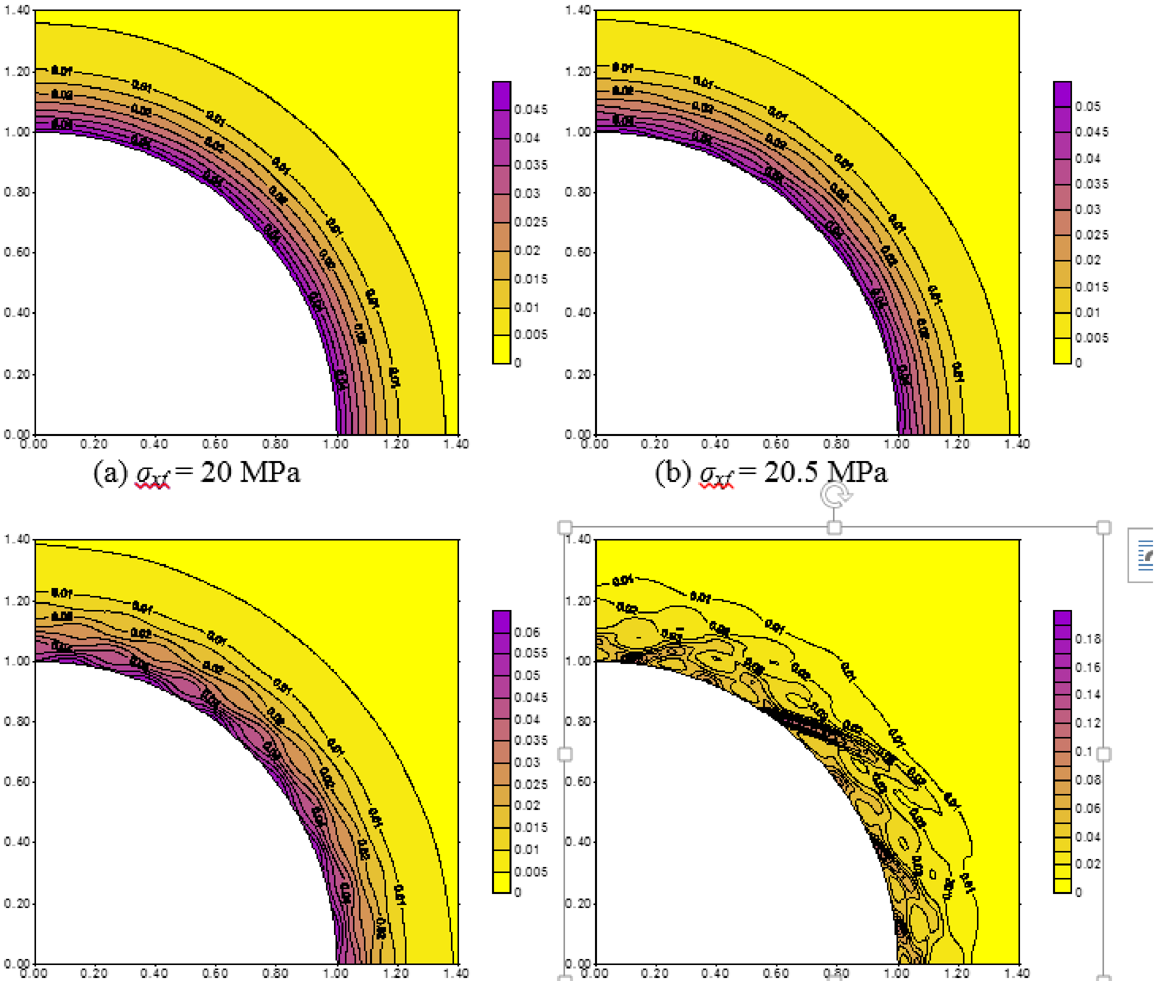




- Warping of hole surface develops as (a) spalling or (b) shear banding

Post-bifurcation numerical simulations

(Papamichos 2010)

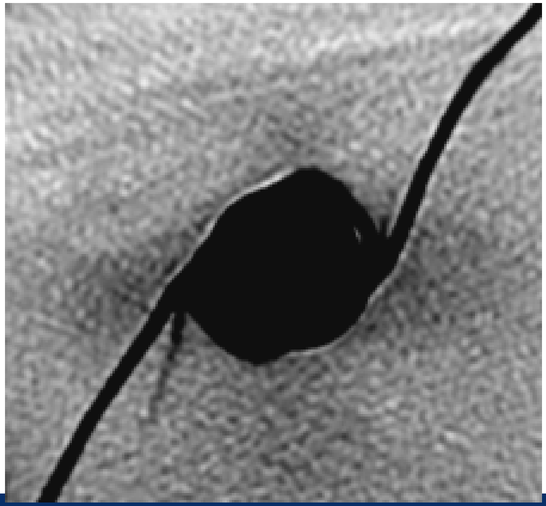


(a) $\sigma_{xf} = 20$ MPa

(b) $\sigma_{xf} = 20.5$ MPa

(c) $\sigma_{xf} = 21$ MPa

(d) $\sigma_{xf} = 22$ MPa



- Non trivial solution $du_{non\_trivial} = du_{trivial} + d\hat{u}$ (or $d\varepsilon_{non\_trivial} = d\varepsilon_{trivial} + d\dot{\varepsilon}$)

- Both $du_{non\_trivial}$ and $du_{trivial}$ must satisfy the same equilibrium b.c. (e.g. loading {t})

- For the incremental form of the virtual work equation this means

$$\int_V \{\delta d\varepsilon\}^T \{d\sigma_{trivial}\} dV = \int_V \{\delta d\varepsilon\}^T \left[C^{ep}\right]\{d\varepsilon_{trivial}\} dV = \int_{\partial V_\sigma} \{\delta du\}\{t\} dS$$

$$\int_V \{\delta d\varepsilon\}^T \{d\sigma_{non\_trivial}\} dV = \int_V \{\delta d\varepsilon\}^T \left[C^{ep}\right]\{d\varepsilon_{non\_trivial}\} dV = \int_{\partial V_\sigma} \{\delta du\}\{t\} dS$$

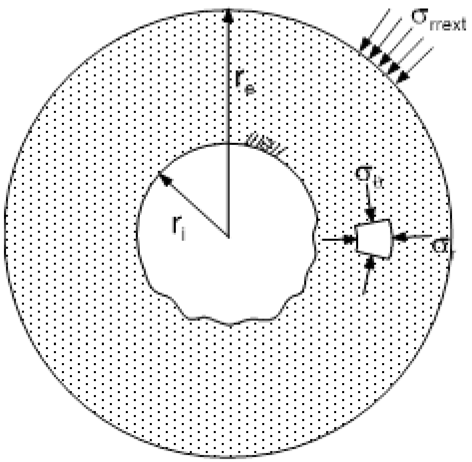
- Then $d\dot{\varepsilon}$ (and $d\hat{u}$) satisfies a condition of zero additional loading or the homogeneous system

$$\int_V \{\delta d\varepsilon\}^T \{d\hat{\sigma}\} dV = \int_V \{\delta d\varepsilon\}^T \left[C^{ep}\right]\{d\hat{\varepsilon}\} dV = 0$$

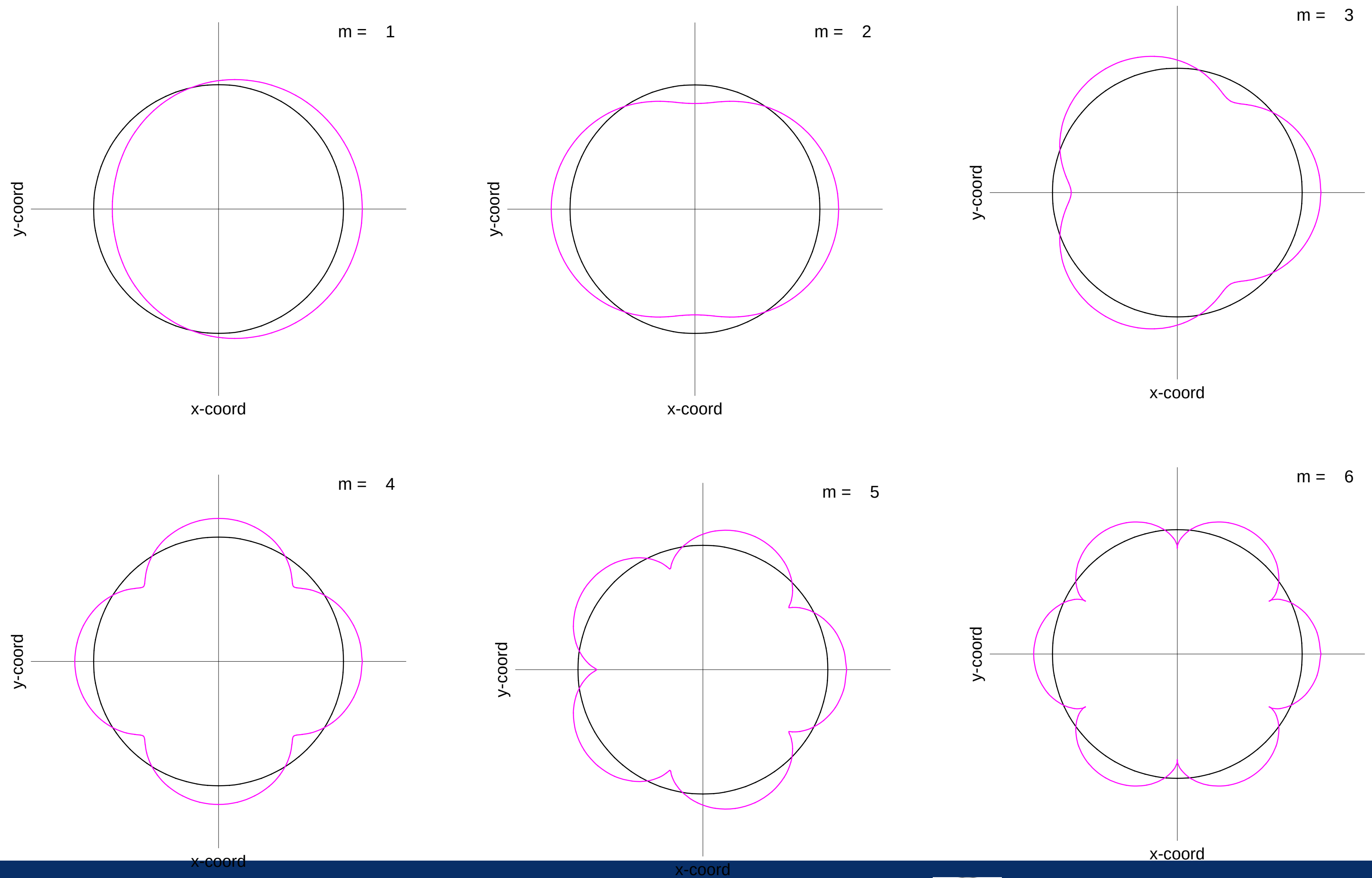
- Non trivial solution for lateral hole failure

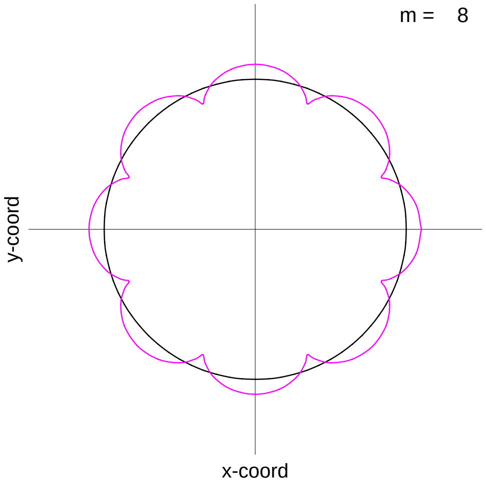
$$
\begin{aligned}
&d\hat{u}_r\left(r,\theta\right)=V_r\left(r\right)\cos\left(m\theta\right) , \quad && d\hat{\omega}_z^c\left(r,\theta\right)=W_z\left(r\right)\sin\left(m\theta\right) \\
&d\hat{u}_\theta\left(r,\theta\right)=V_\theta\left(r\right)\sin\left(m\theta\right) , \quad && d\hat{\omega}_r^c=0 \\
&d\hat{u}_z=0 , && d\hat{\omega}_\theta^c=0
\end{aligned}
$$

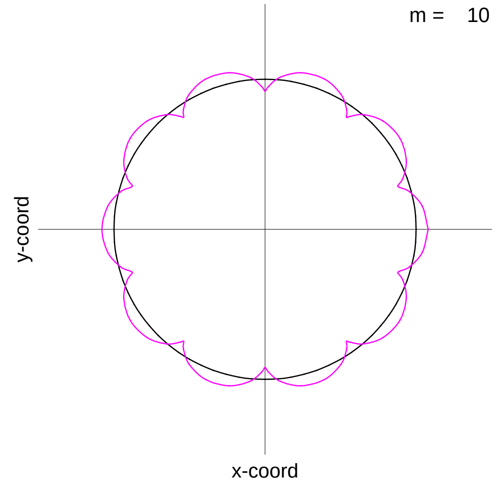
- m = 1, 2, 3, ... is the wavenumber of the warping mode
- Wavelength $W = 2\pi r_i/m$

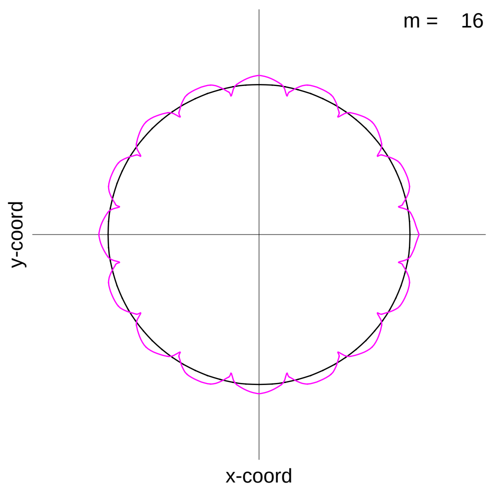


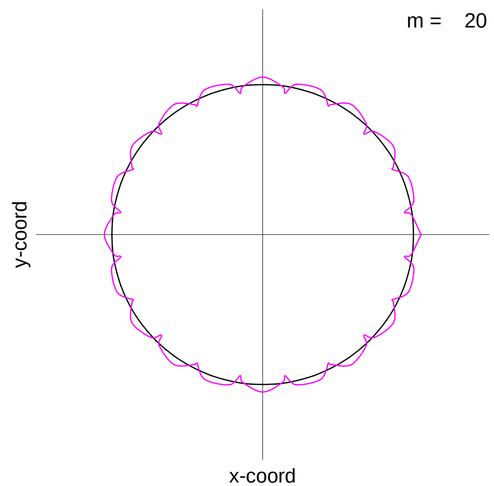
Bifurcation modes




m = 8
y-coord
x-coord


m = 10
y-coord
x-coord


m = 16
y-coord
x-coord


m = 20
y-coord
x-coord

- Shape functions relate displacements to nodal values $\{d\hat{u}\} = [N]\{dU\}$
- Strain – displacement differential operator $\{d\hat{\varepsilon}\} = [L]\{d\hat{u}\}$
- Gives finally $\{d\hat{\varepsilon}\} = [B]\{dU\}$

$$[B_i] = \begin{bmatrix} N_{i,r}\cos(m\theta) & 0 & 0 \\ \frac{1}{r}N_i\cos(m\theta) & \frac{m}{r}N_i\cos(m\theta) & 0 \\ -\frac{m}{r}N_i\sin(m\theta) & -\frac{1}{r}N_i\sin(m\theta) & N_i\sin(m\theta) \\ 0 & N_{i,r}\sin(m\theta) & -N_i\sin(m\theta) \\ 0 & 0 & RN_{i,r}\sin(m\theta) \\ 0 & 0 & \frac{mR}{r}N_i\cos(m\theta) \end{bmatrix} = [B_i^s]\sin(m\theta) + [B_i^c]\cos(m\theta)$$

- Substitution into the virtual work eqn $$\int_V \{\delta d\varepsilon\}^T \left[C^{ep}\right]\{d\hat{\varepsilon}\} dV = 0$$
- Gives $$\int_V [B]^T \left[\hat{C}^{ep}\right][B] dV \{dU\} = 0$$
- Where $\left[\hat{C}^{ep}\right]$ contains only the relevant terms of $\left[C^{ep}\right]$
- We can simplify by direct integration over θ and *z* to obtain

$$[K]\{dU\} = 0\,, \qquad [K] = \int_{r_i}^{r_e} \left\{\left[B^s\right]^T \left[\hat{C}^{ep}\right]\left[B^s\right] + \left[B^c\right]^T \left[\hat{C}^{ep}\right]\left[B^c\right]\right\} r dr$$

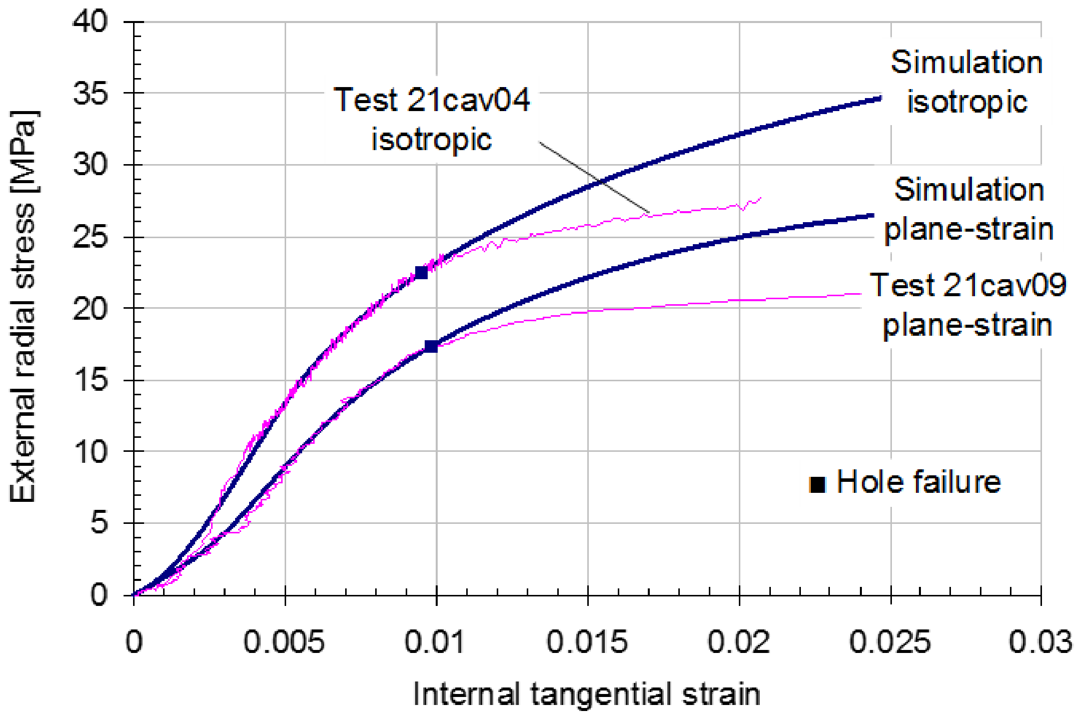
- Within a FE scheme, the solution to this eigenvalue problem is obtained by requiring that the global stiffness matrix [*K*] becomes singular, i.e.
- Bifurcation condition: $$\det[K] = 0$$

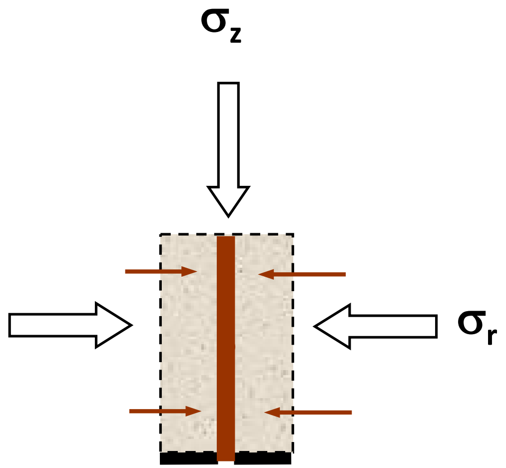
Comments

- Reduction to an 1-d problem
- Each node has three degrees of freedom. These are the radial V_r and tangential V_θ displacement amplitude and the Cosserat microrotation W_z amplitude
- The bifurcation problem requires that the **global** stiffness matrix becomes singular
- In inhomogeneous problems like the present, failure of local stability conditions does not necessarily imply loss of uniqueness
- In fact, the obtained bifurcation points correspond to loading states where elements close to the hole have entered the softening regime. Although at these elements the local stability criterion is violated, the global stiffness matrix remains positive.
- The loading stresses at which the bifurcation condition is satisfied depend on the wavenumber m of the bifurcation mode. There exists a critical wavenumber m_{cr} that corresponds to the least required load.
- It is obtained by solving the bifurcation condition for various m and selecting the m_{cr} corresponding to the least required load. It maybe, therefore, assumed that the hole would fail under m_{cr}.

Simulation of hollow-cylinder experiments

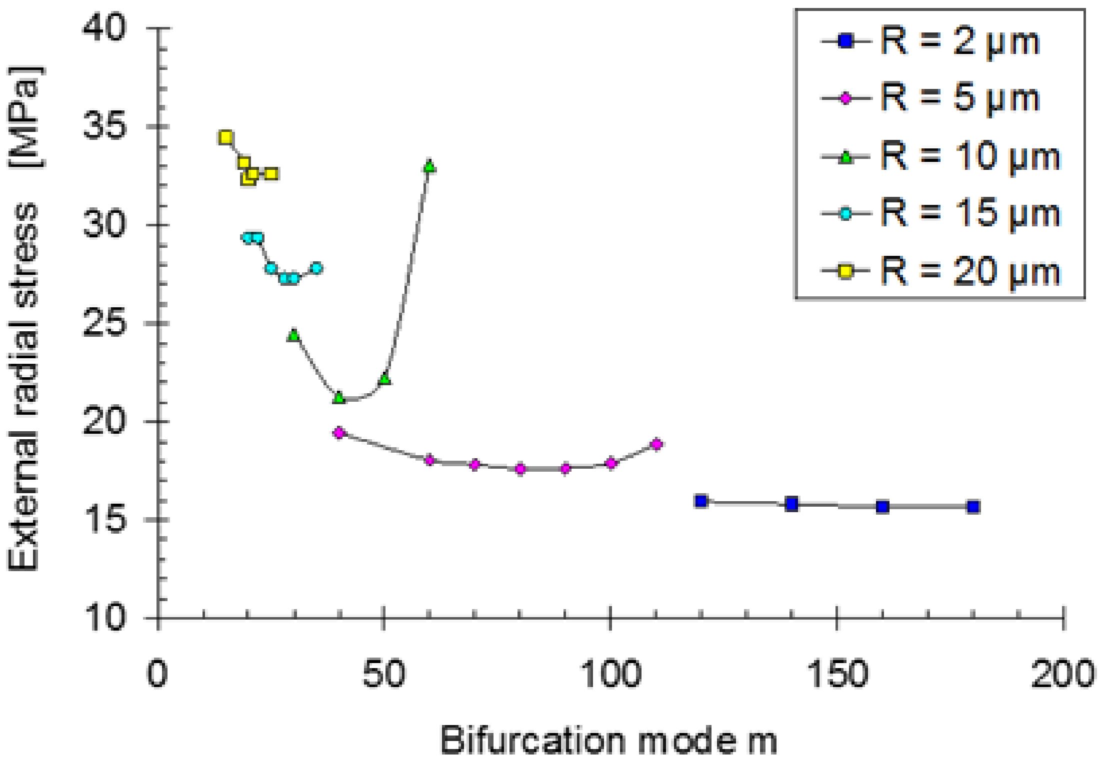
- Hollow cylinder tests under isotropic and plane-strain loading
- Identification of onset of bifurcation





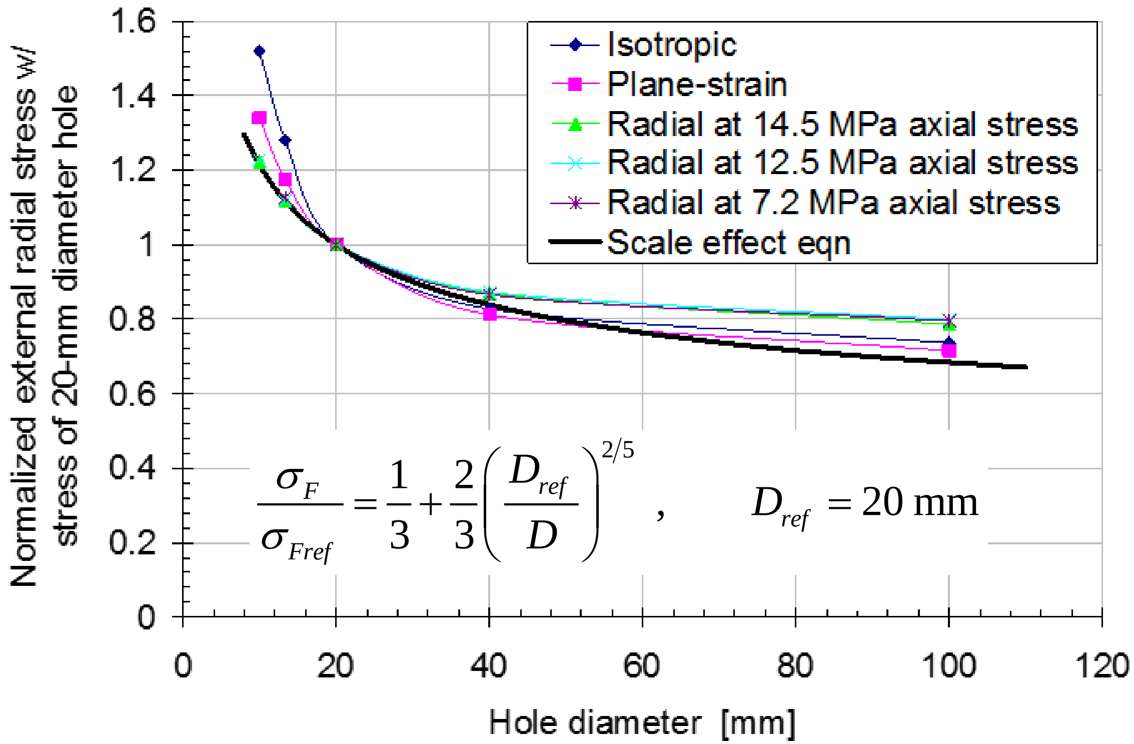
- If the internal length *R* is known we can make forward prediction
  If *R* is <u>NOT</u> known, calibrate in one test and forward predict the other tests

- The minimum in each curve leads to the selection of the critical mode m_{cr} corresponding to the least external radial stress

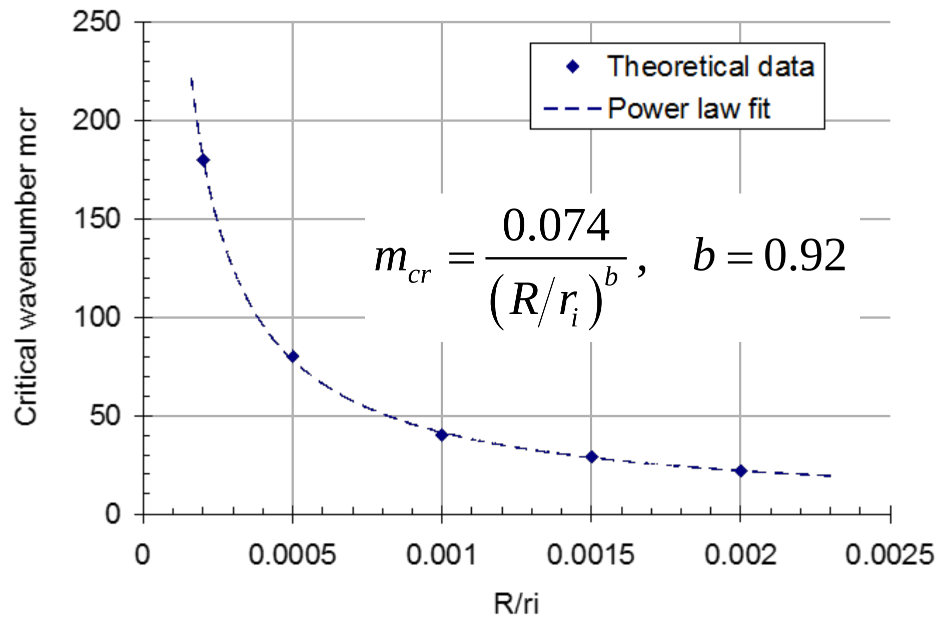


Size effect of the failure stress

- Vary internal radius r_i to obtain size effect (scales with R)



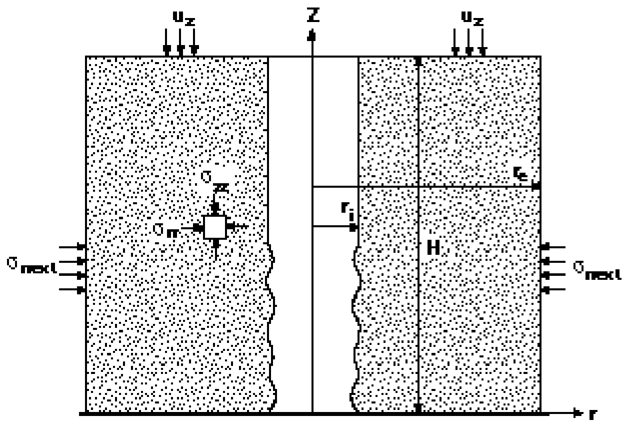
- m_{cr} decreases with R/r_i
  - *Large holes -> large wavenumber*



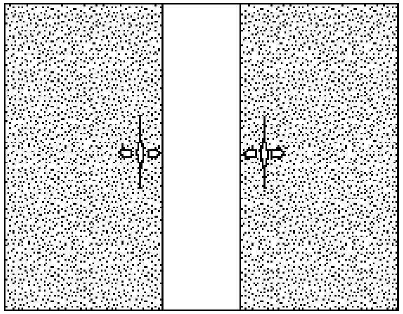
$$m_{cr} = \frac{0.074}{(R/r_i)^b}, \quad b = 0.92$$

- Experimental and numerical evidence show that although localization may initiate under a high mode, a lower mode may finally evolve
- Indeed experiments show that an initial mode m = 6 localization evolves to an apparent mode m = 3 failure [Haimson 1993]

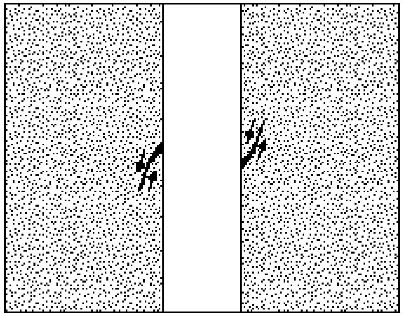
Axial failure



c



d



- Axial warping of hole surface develops as (c) spalling or (d) shear banding

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Axial failure

- Non trivial solution for axial hole failure

$$d\hat{u}_r(r,z) = V_r(r)\cos(m\pi z/H)$$

$$d\hat{u}_\theta = 0$$

$$d\hat{u}_z(r,z) = V_z(r)\sin(m\pi z/H)$$

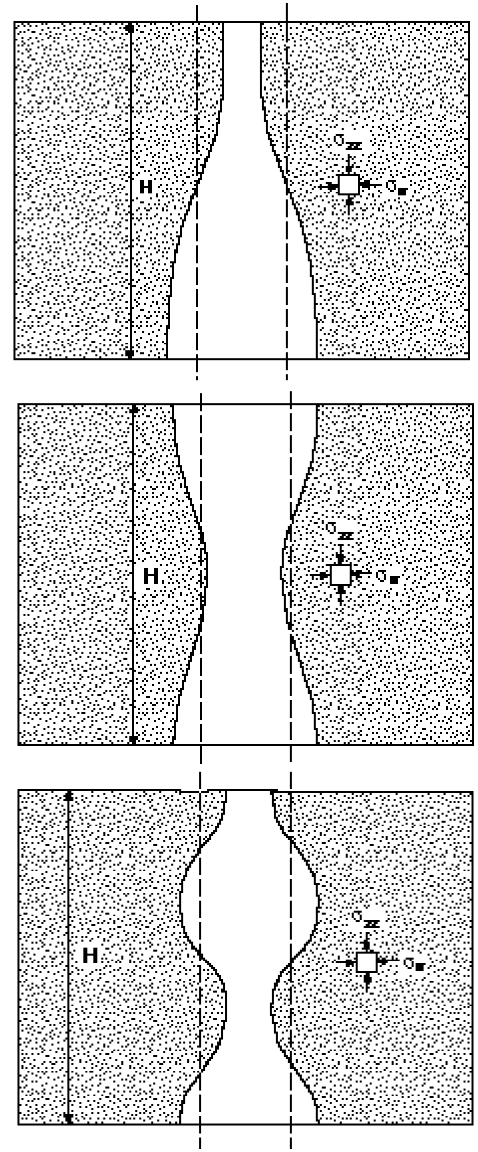
$$d\hat{\omega}_z^c = d\hat{\omega}_r^c = 0$$

$$d\hat{\omega}_\theta^c(r,z) = W_\theta(r)\sin(m\pi z/H)$$

m = 1

m = 2

m = 3



- Following the same FE discretization, the solution to this eigenvalue problem is obtained by requiring that the global stiffness matrix [*K*] becomes singular, i.e.

$$[K]\{dU\}=0\,,\qquad [K]=\int_{r_i}^{r_e}\left\{\left[B^s\right]^T\left[\tilde{C}^{ep}\right]\left[B^s\right]+\left[B^c\right]^T\left[\tilde{C}^{ep}\right]\left[B^c\right]\right\}rdr$$

- Bifurcation condition: $\det\left[K\right]=0$

- Where ξ = mπ/H

$$\left[B_i\right]=\left[B_i^s\right]\sin(\xi z)+\left[B_i^c\right]\cos(\xi z)=\begin{bmatrix} N_{i,r}\cos(\xi z) & 0 & 0 \\ \frac{1}{r}N_i\cos(\xi z) & 0 & 0 \\ 0 & \xi N_i\cos(\xi z) & 0 \\ 0 & N_{i,r}\sin(\xi z) & N_i\sin(\xi z) \\ -\xi N_i\sin(\xi z) & 0 & -N_i\sin(\xi z) \\ 0 & 0 & -\frac{R}{r}N_i\sin(\xi z) \\ 0 & 0 & RN_{i,r}\sin(\xi z) \\ 0 & 0 & \xi RN_i\cos(\xi z) \end{bmatrix}$$

2. Folding of elastic/viscoelastic layered media as a bifurcation problem

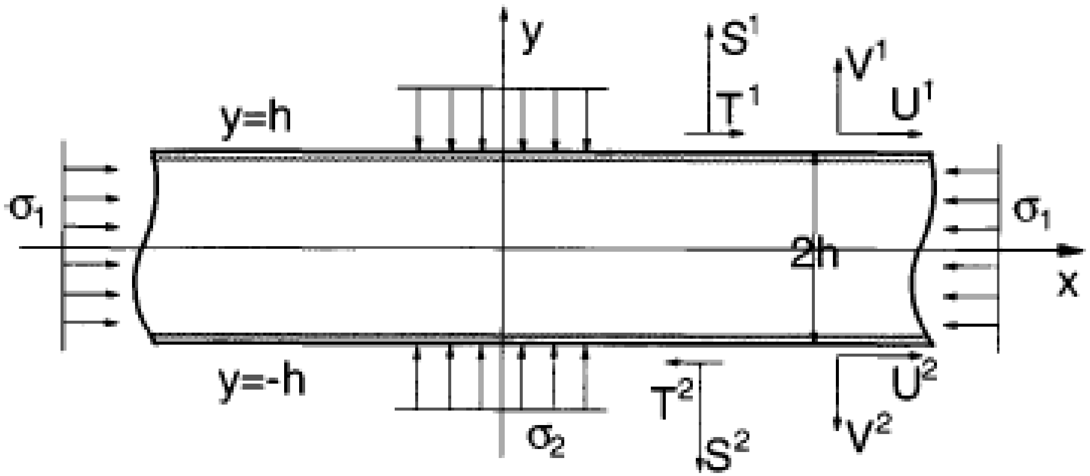
- Constitutive equations of large strain elasticity theory are used to study buckling of elastic layered media
  - Geometric non-linearity
- Buckling modes can explain various periodic structures in geology such as folds

- Maurice Biot has presented an analysis of folding of stratified sedimentary rock in a series of pioneering papers and in his book "Mechanics of incremental deformations" (1965)

- The folding mechanism considered in Biot's theory is the spontaneous folding caused by instability under a compressive load acting in a direction parallel to the layers
- From the geological viewpoint, a purely elastic theory is not sufficient to explain folding
  - Time-dependent phenomena such as viscous behavior must be taken into account
- Biot (1957) developed a general theory of folding of a compressed viscoelastic layer embedded in an infinite medium of another viscoelastic material
  - In general, there exists a lower and a higher critical load between which folding occurs at a finite rate with a dominant wavelength
  - This is the wavelength whose amplitude increases at the fastest rate
  - An experimental verification of Biot's theory of folding of stratified viscoelastic media in compression is presented by Biot et al. (1961)

Buckling of a layer under initial stress

- Layer of thickness 2h under in situ stress σ_1 , σ_2
- Do they exist other solutions superimposed on the large strain uniform compression?
  - They will satisfy homogeneous boundary conditions



Large strains/deformations

- C = initial configuration
- $\bar{C}$ = Current configuration
- C* = Resulting configuration

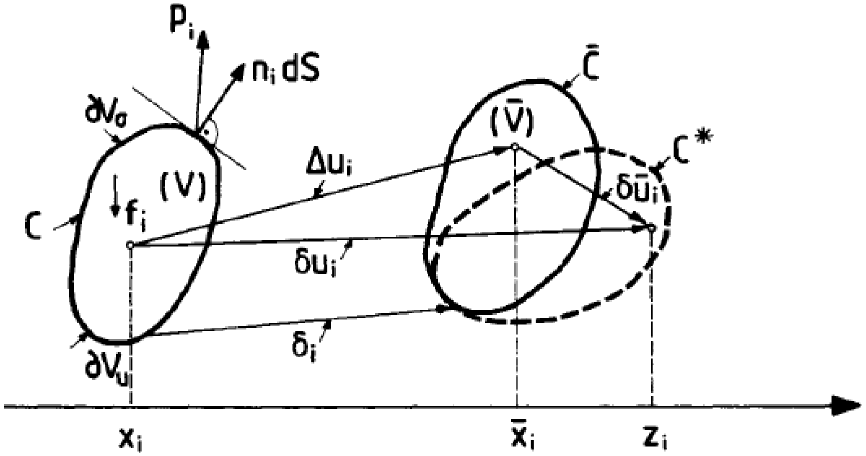
- σ_{ij} = stresses in C
- $\bar{\sigma}_{ij}$ = stresses in $\bar{C}$

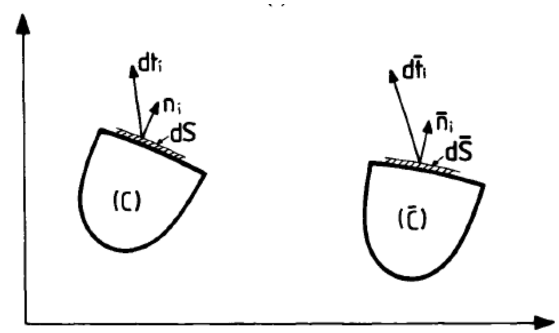
- Stress vectors in initial and current configuration

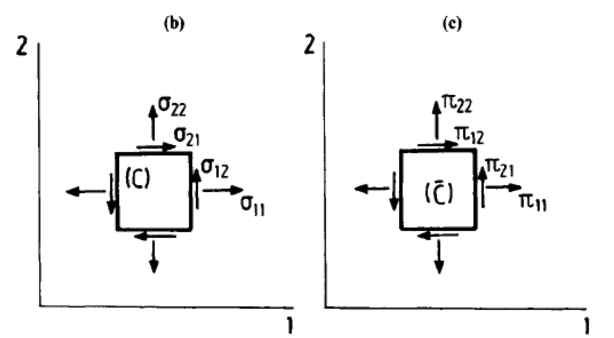
$$dt_i = \sigma_{ij} n_i \left(dS \right)$$

$$d\bar{t}_i = \bar{\sigma}_{ij} n_i \left(d\bar{S} \right)$$

$$d\bar{t}_i = \pi_{ij} n_j \left(dS \right)$$





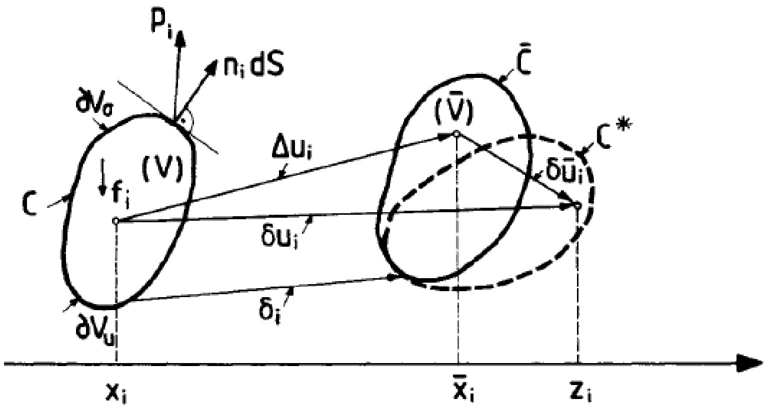


- σ_{ij} = Cauchy stress tensor (symmetric)
- π_{ij} = 1st Piola-Kirchoff stress tensor (non-symmetric)

$$\Delta\pi_{ij} = \pi_{ij} - \sigma_{ij} = \Delta T_{ij} + \Delta\omega_{ik}\sigma_{kj} - \sigma_{ik}\Delta\varepsilon_{kj} + \sigma_{ij}\Delta\varepsilon_{kk}$$

- ΔT_{ij} = objective part = Jaumann stress increment of the Cauchy stress
- $\Delta\omega_{ik}\sigma_{kj} - \sigma_{ik}\Delta\varepsilon_{kj} + \sigma_{ij}\Delta\varepsilon_{kk}$ = geometric part
- Jaumann stress increment is related to strain increments through hypoelastic laws (constitutive)

- Jaumann stress increment is related to strain increments through hypoelastic laws (constitutive) for anisotropic materials in plane-strain



$$dT_{11} = C_{11} d\varepsilon_{11} + C_{12} d\varepsilon_{22}$$
$$dT_{22} = C_{21} d\varepsilon_{11} + C_{22} d\varepsilon_{22}$$
$$dT_{12} = 2G d\varepsilon_{12}$$

- Incemental equilibrium equations $d\pi_{ij,j} = 0$
- Incremental boundary conditions $d\pi_{ij} n_j = d\pi_i$

from the current configuration C_bar to the resulting configuration C* are written in terms of the 1st P.K. stress

- Substitution into the equilibrium equations

$$C_{11}du_{1,11}+\left(G-\tau\right)du_{1,22}+\left(C_{12}+G+\tau\right)du_{2,12}=0$$
$$\left(C_{21}+G-\tau\right)du_{1,12}+\left(G+\tau\right)du_{2,11}+C_{22}du_{2,22}=0$$

- where $\tau=\left(\sigma_1-\sigma_2\right)/2$

- Assumed displacement to be given in terms of two unknown amplitude functions

$$du_1=U\left(y\right)\sin\left(\beta x\right),\qquad U\left(y\right)=Ae^{\gamma y}$$
$$du_2=V\left(y\right)\cos\left(\beta x\right),\qquad V\left(y\right)=Be^{\gamma y}$$

$$x=x_1/L \qquad y=x_2/L$$
$$W=2\pi L/\beta$$

- Substitution in the equilibrium equations gives

$$A\left[-C_{11}+(G-\tau)Z^2\right]+B\left[-(C_{12}+G+\tau)Z\right]=0$$
$$A\left[(C_{21}+G-\tau)Z\right]+B\left[-(G+\tau)+C_{22}Z^2\right]=0$$

$$Z=\gamma/\beta$$

- For non-trivial (non-zero) solutions for the displacements : det = 0

$$aZ^4-bZ^2+c=0$$

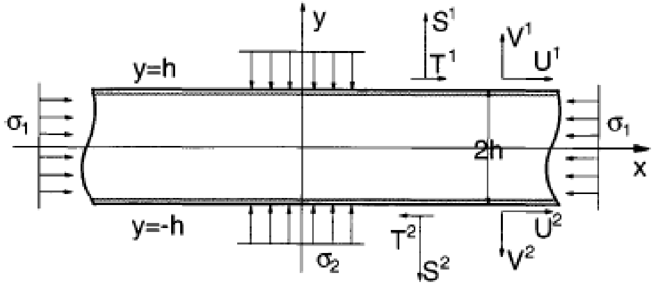
$$a=C_{22}(G-\tau)$$

$$b=C_{11}C_{22}-C_{12}C_{21}-C_{12}(G-\tau)-C_{21}(G+\tau)$$

$$c=C_{11}(G+\tau)$$

- Four roots give four solutions: Complete solution is a linear combination of the solutions w/ a_k integration constants

$$U(y)=\sum_{k=1}^{4} a_k U_k\ , \qquad V(y)=\sum_{k=1}^{4} a_k V_k$$



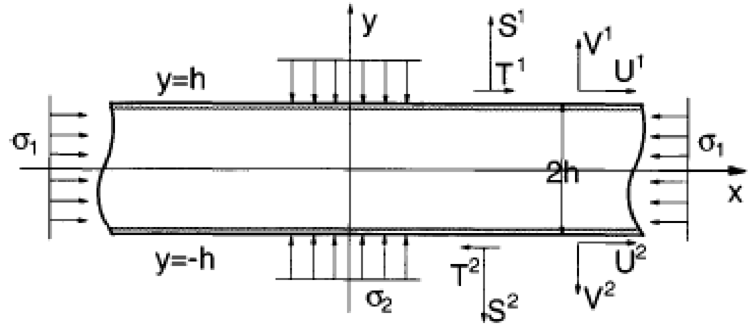
- Tractions $d\pi_i = d\pi_{ij} n_j$ are

$$d\pi_1 = \frac{\beta}{L} T(y)\sin(\beta x), \quad d\pi_2 = \frac{\beta}{L} S(y)\cos(\beta x)$$

- where $S(y)=\sum_{k=1}^{4} a_k S_k\ , \quad S_k = (C_{21}-\tau-p)U_k + C_{22}V_k'/\beta$

$$T(y)=\sum_{k=1}^{4} a_k T_k\ , \quad T_k = (G-\tau)U_k'/\beta + (G-p)V_k$$

At layer boundaries



- Eigendisplacements and tractions

$$U^1 = \sum_{k=1}^{4} a_k U_k\left(h/L\right), \quad V^1 = \sum_{k=1}^{4} a_k V_k\left(h/L\right)$$

$$U^2 = \sum_{k=1}^{4} a_k U_k\left(-h/L\right), \quad V^2 = \sum_{k=1}^{4} a_k V_k\left(-h/L\right)$$

$$S^1 = \sum_{k=1}^{4} a_k S_k\left(h/L\right), \quad T^1 = \sum_{k=1}^{4} a_k T_k\left(h/L\right)$$

$$S^2 = \sum_{k=1}^{4} a_k S_k\left(-h/L\right), \quad T^2 = \sum_{k=1}^{4} a_k T_k\left(-h/L\right)$$

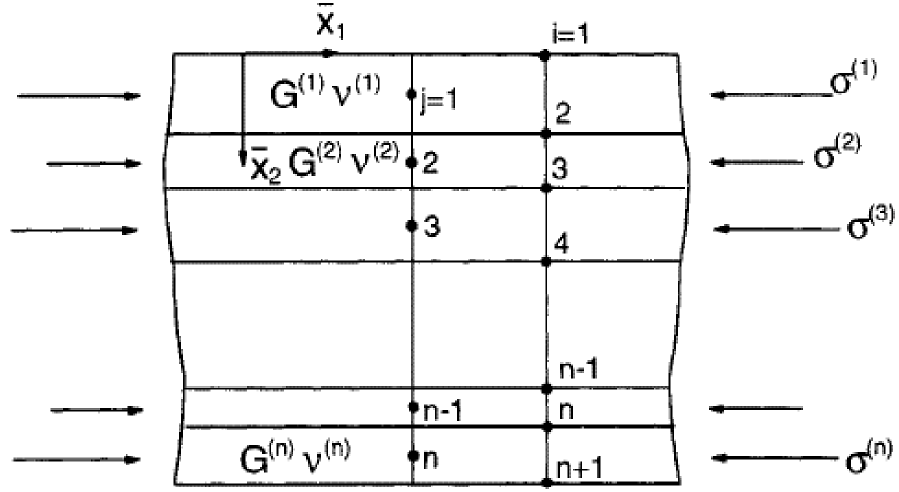
In matrix form solution for 1 layer

- For i = 1 upper layer and i = 2 lower layer

$$\begin{Bmatrix} U^i \\ V^i \\ S^i \\ T^i \end{Bmatrix} = \begin{bmatrix} U_1^i & U_2^i & U_3^i & U_4^i \\ V_1^i & V_2^i & V_3^i & V_4^i \\ S_1^i & S_2^i & S_3^i & S_4^i \\ T_1^i & T_2^i & T_3^i & T_4^i \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}$$

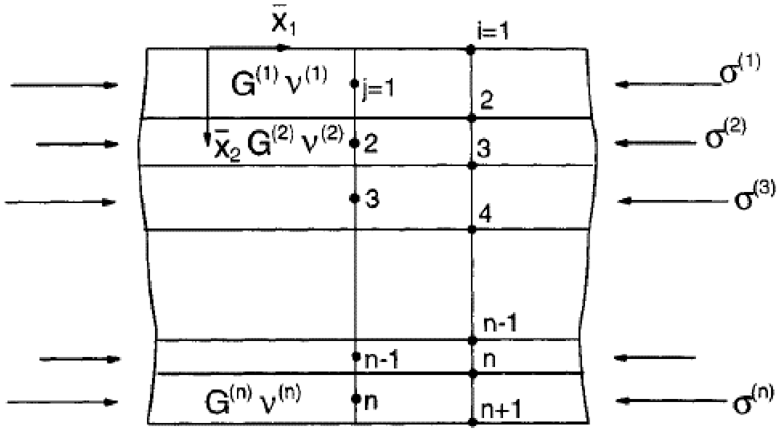
Buckling of a layer system—the transfer matrix technique

- System of *n* layers of different materials and different initial stresses parallel to the layer axis with a global coordinate system located at the top layer

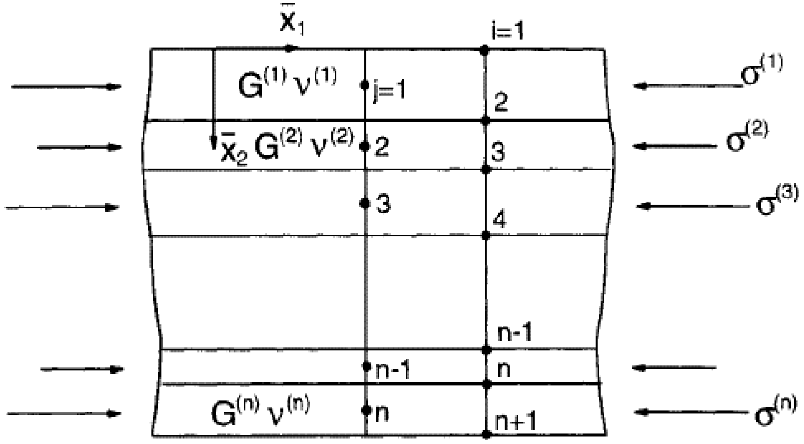


- Assuming perfect adherence at the interfaces
  - Incremental stresses and displacements are continuous along the interfaces

- Amplitude of the incremental stresses and displacements for the i^{th} interface of the j^{th} layer



- $$\begin{Bmatrix} U^{ij} \\ V^{ij} \\ S^{ij} \\ T^{ij} \end{Bmatrix} = \begin{bmatrix} U_1^{ij} & U_2^{ij} & U_3^{ij} & U_4^{ij} \\ V_1^{ij} & V_2^{ij} & V_3^{ij} & V_4^{ij} \\ S_1^{ij} & S_2^{ij} & S_3^{ij} & S_4^{ij} \\ T_1^{ij} & T_2^{ij} & T_3^{ij} & T_4^{ij} \end{bmatrix} \begin{Bmatrix} a_1^j \\ a_2^j \\ a_3^j \\ a_4^j \end{Bmatrix}$$ OR $$\{X^{ij}\} = [F^{ij}]\{A^j\}$$

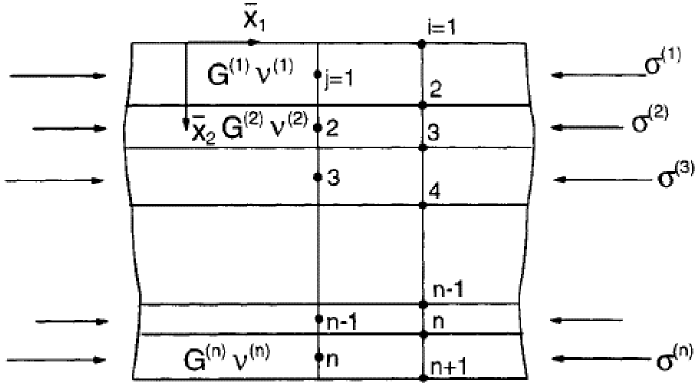


- Continuity of the incremental displacements and tractions at all interfaces, the integration constants of every layer are linked to the integration constants of the top layer as follows

$$\left\{A^n\right\}=\left[F\right]\left\{A^1\right\},\quad \left[F\right]=\left[F^n\right]\cdots\left[F^2\right],\quad \left[F^k\right]=\left[F^{kk}\right]^{-1}\left[F^{k(k-1)}\right]$$

- In order to formulate the eigenvalue problem we have to consider boundary conditions only at the upper and lower boundary surfaces of the layered medium

Example



- Zero tractions at the upper boundary surface (*i* = 1, *j* = 1) and zero displacements at the lower boundary surface (*i* = *n*+1, *j* = *n*)

$$\left[Y^1\right]\left\{A^1\right\}=0\,,\quad \left[Y^n\right]\left\{A^n\right\}=0 \qquad\qquad \left\{A^n\right\}=\left[F\right]\left\{A^1\right\}$$

- $\left[Y^1\right]$ = 2 last rows of $\left[F^{11}\right]$
- $\left[Y^n\right]$ = 2 first rows of $\left[F^{(n+1)n}\right]$

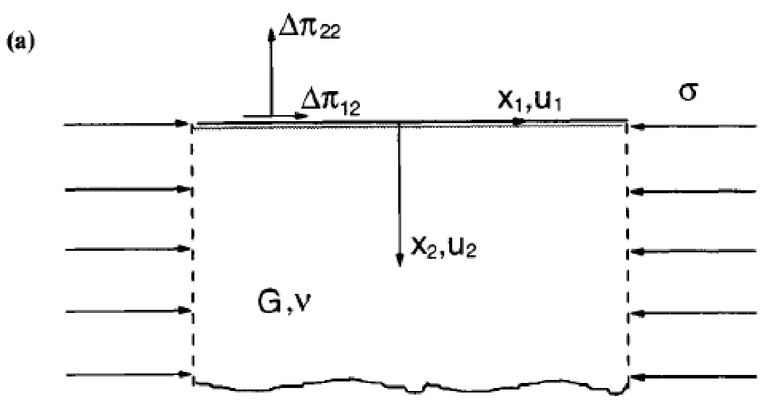
- Homogeneous algebraic system of equations for the integration constants

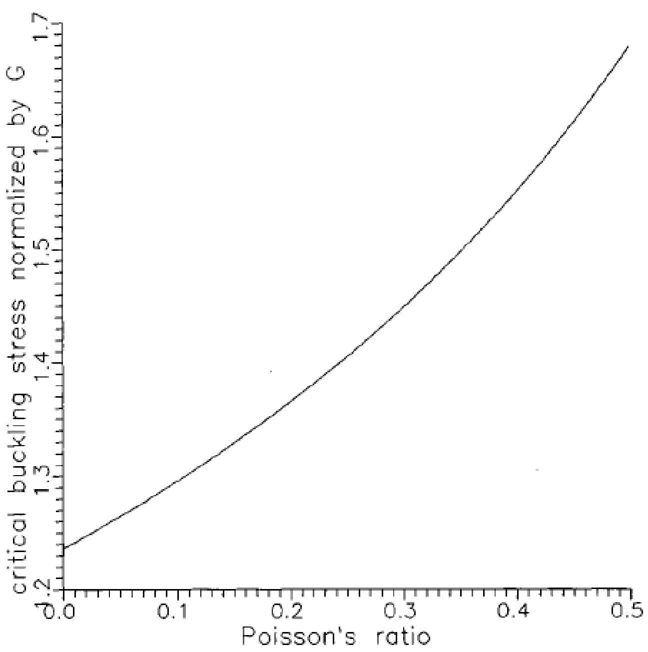
$$\begin{bmatrix} \left[Y^n\right] & \left[F\right] \\ \left[Y^1\right] & \end{bmatrix}\left\{A^1\right\}=0 \qquad \text{or} \qquad \left[Y\right]\left\{A^1\right\}=0$$

- For non-trivial solutions $\det\left(\left[Y\right]\right)=0$
- The least load σ_1 that satisfies the bifurcation condition

Buckling of homogeneous half space

- The buckling condition for a homogeneous half-space is independent of the wavelength of the considered mode
  - No length appears in this problem and consequently the various modes corresponding to different wavelengths cannot be differentiated

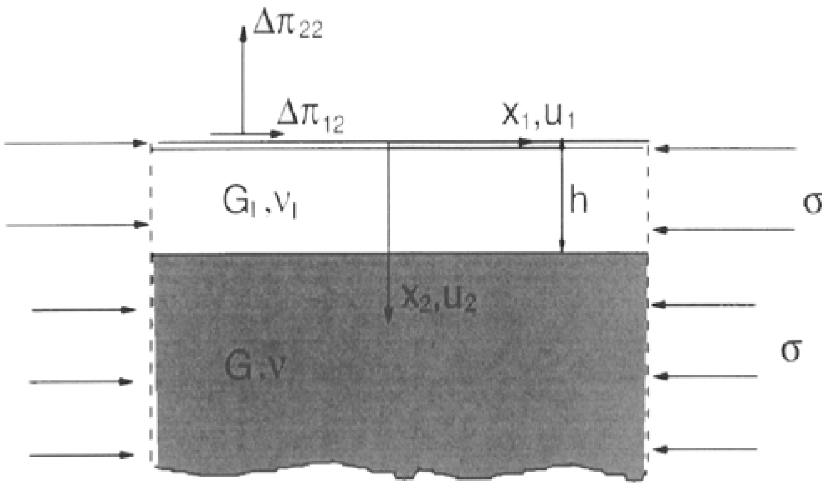




Buckling of layer on a half space

- Introduce length to select a particular buckling mode

- An example is the buckling of a layer with height *h* on top of a half-space due to a horizontal homogeneous strain field

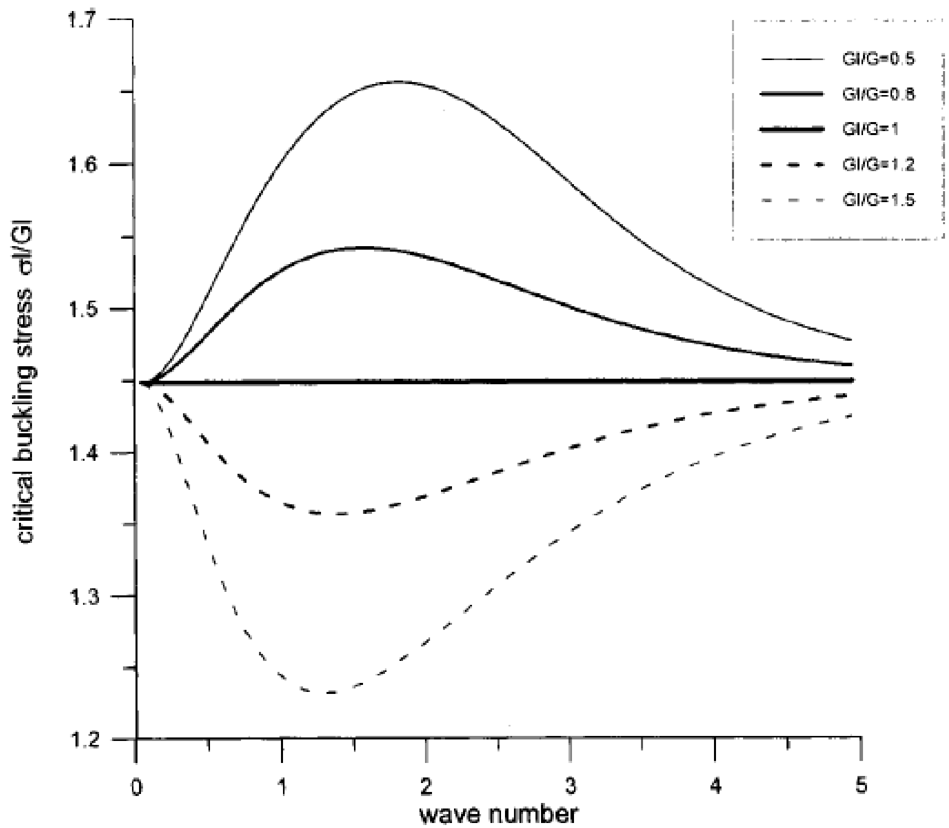


Critical buckling stress vs wavenumber $\beta = 2\pi h/W$

- GL/G = 0.5 – 1.5

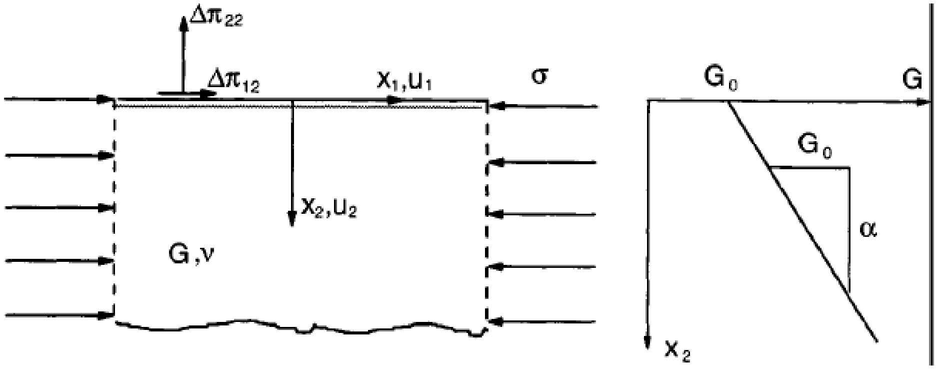
- For $\beta \rightarrow 0$ or $\rightarrow \infty$
- $\sigma_{cr} = \sigma_{cr\_half\_space}$

- For stiffer layer at the top a dominant wavelength is obtained

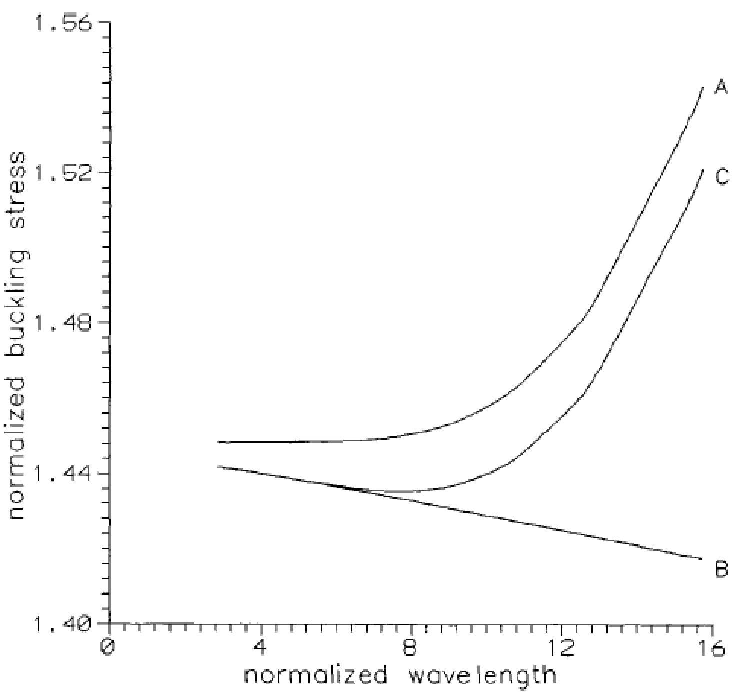


Buckling of a half-space under geostatic compression

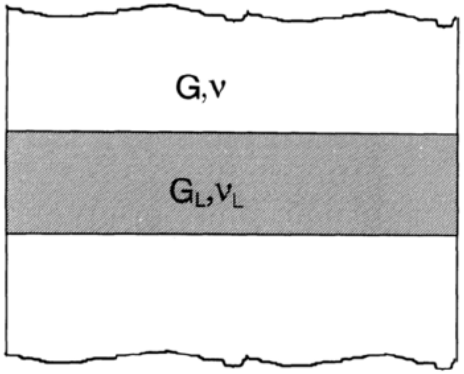
- (A) Buckling of a layer on a rigid base under constant horizontal load: $W_{cr} \rightarrow 0$
- (B) Buckling of a half-space under horizontal load increasing with depth $W_{cr} \rightarrow \infty$
- (C) Buckling of a layer under horizontal load increasing with depth W_{cr} = selected
- Two competing factors lead to wavelength selection

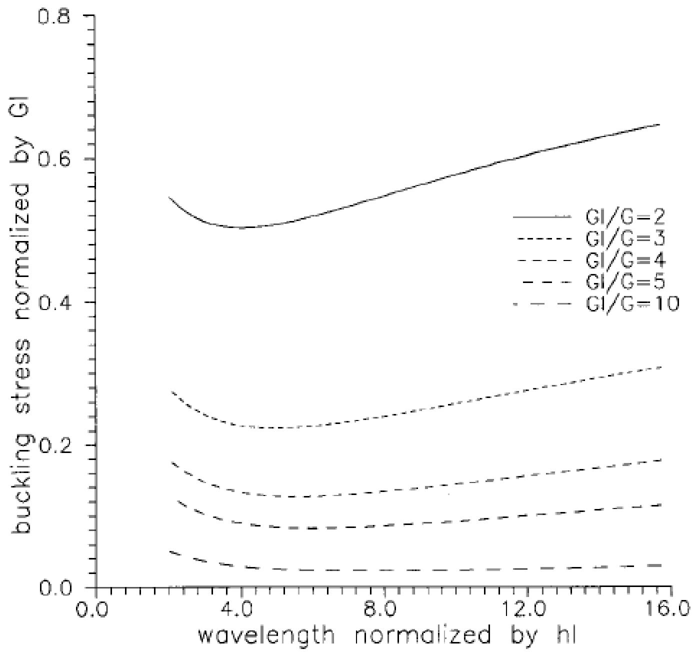


Gibson half-space under horizontal compression σ.



Interfacial instability

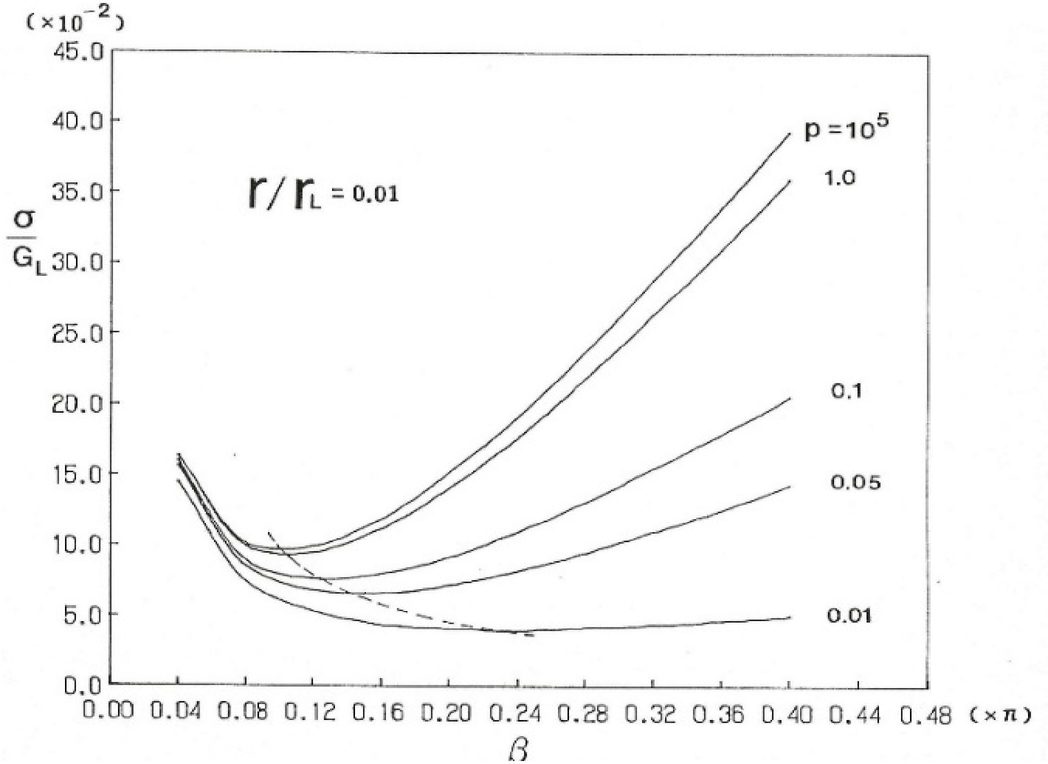




- Buckling load for stiff layer
- Symmetric mode only

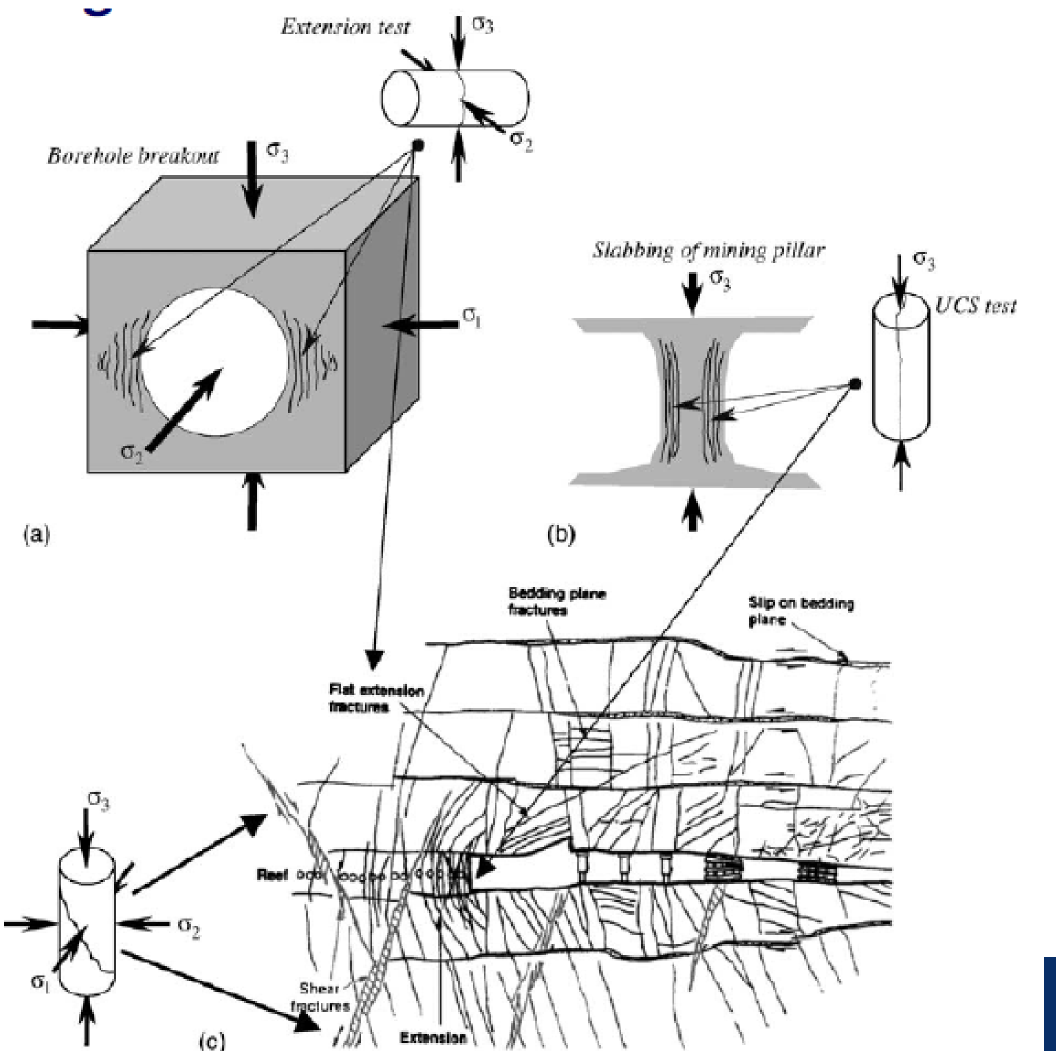
Viscoelastic materials

- Exists dominant wavelength for fastest growth at given stress level
- Critical buckling stress of a viscoelastic layer on a viscoelastic half space with relaxation constant ratio r/rL = 0.01 (Maxwell material) and rate of growth *p*.

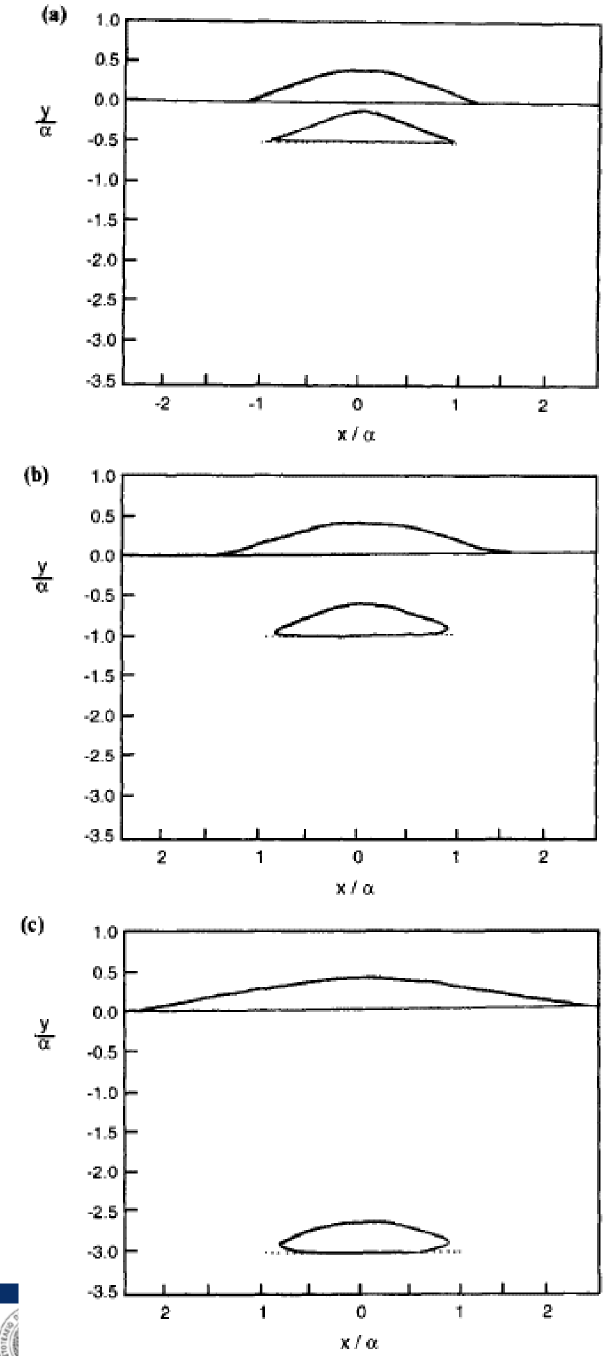


3. Spalling and buckling of surface parallel cracks with application in rock bursting in mining

- Axial splitting in uniaxial loading, spalling of a free surface and rock bursting are common phenomena in brittle materials, like rock and concrete, under compressive stresses parallel to a free surface

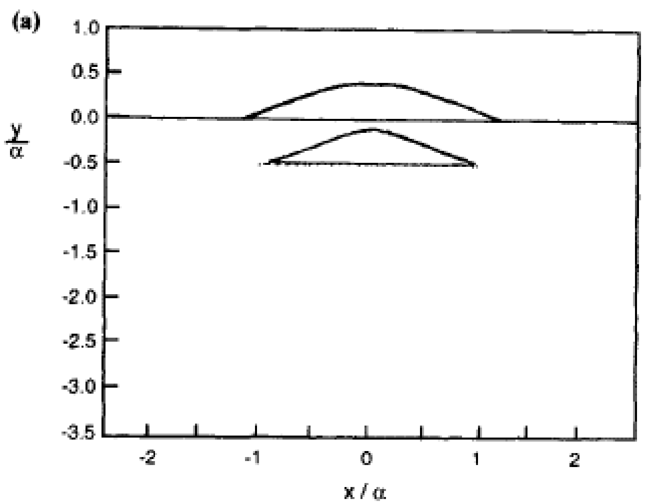


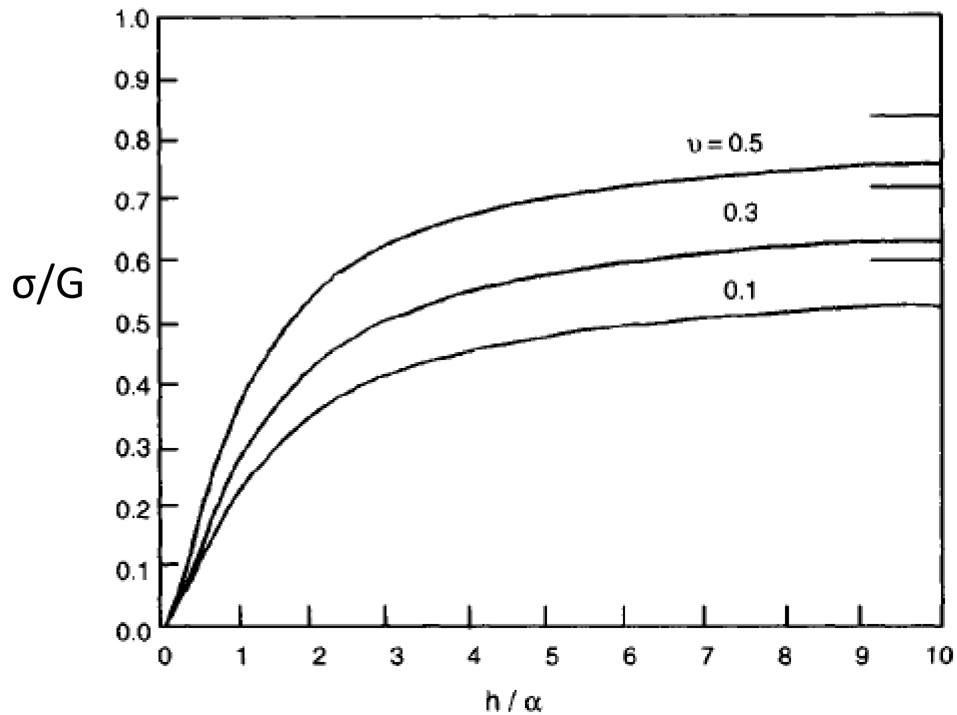
- Axial splitting and spalling as the result of interaction between surface instabilities and surface parallel Griffith cracks
- Surface instabilities in a uniformly stressed half-space, produce secondary tensile stresses, which, for material points close to a free surface remain unbalanced in the direction normal to the surface
- These tensile stresses cause latent, surface parallel cracks to open and thus magnify the effect of diffuse bifurcation
- Tensile stress concentrations develop at the crack tips resulting in unstable crack growth and finally axial splitting and spalling of the material.



Buckling of a half space with a single crack

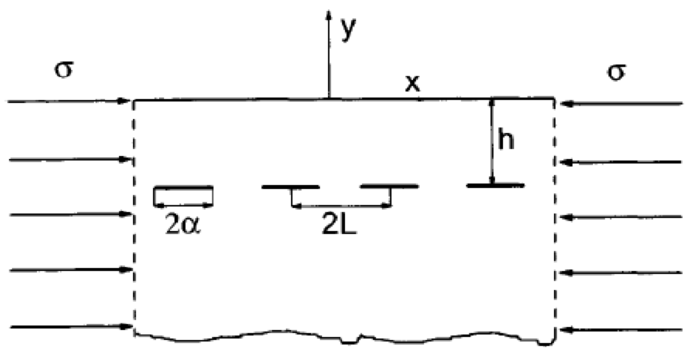
- Analytical solution exists (Nazarenko 1985, 1986)



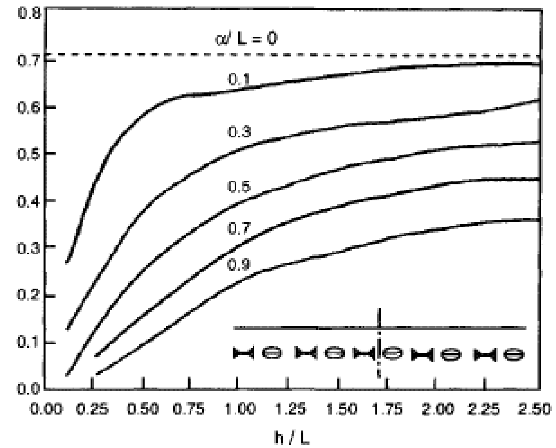


Buckling of half space with arrays of cracks

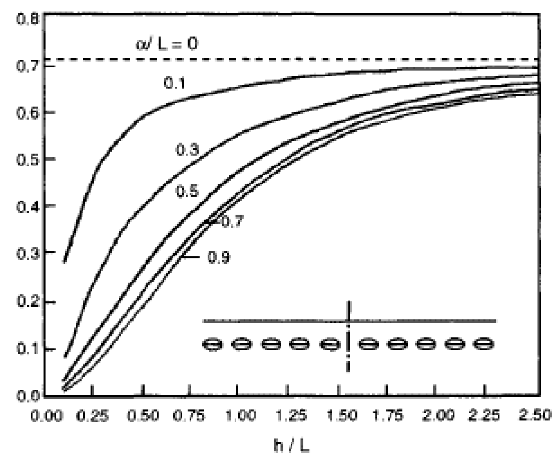
- Distributed damage in the material
- Numerical solution (Vardoulakis + Papamichos 1991) based on Displacment Discontinuity method
- For 1 periodic array of cracks (Keer 1981)



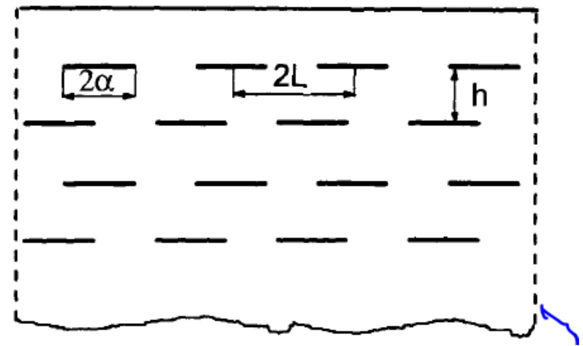
σ/G



σ/G

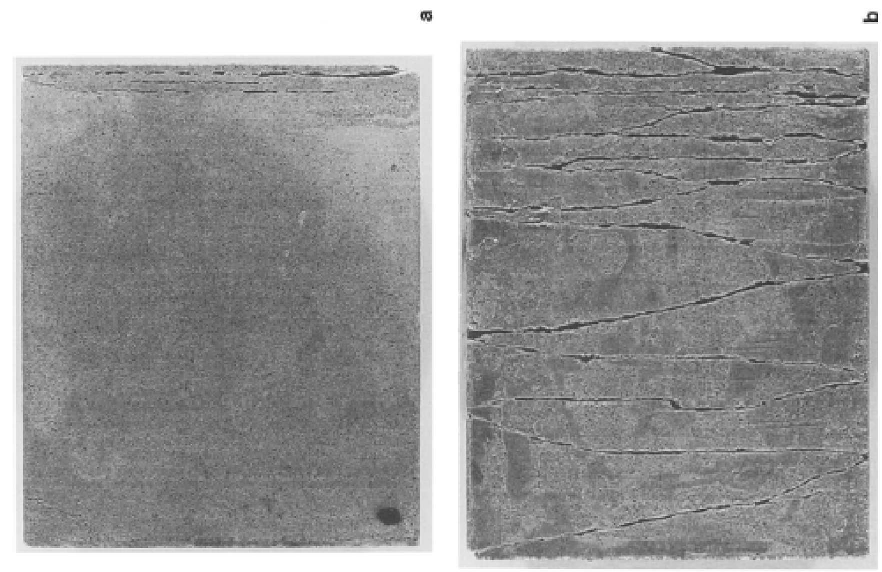


Muliple arrays of cracks



- Eigendispalcements affect primarily the row of cracks

- Progressive spalling that starts close to the free surface and subsequently progresses deeper into the material

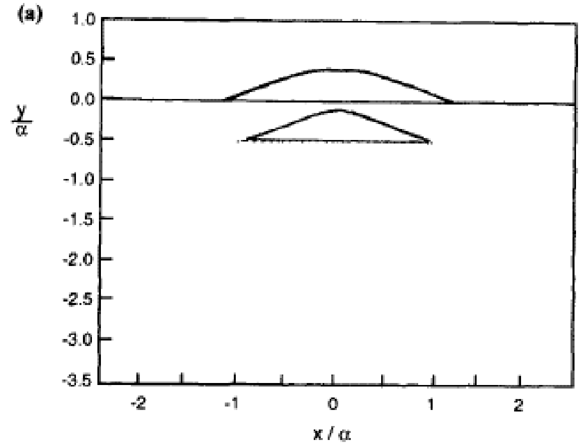


(Papamichos, Labuz, Vardoulakis 1994)

Spalling failure in Berea sandstone specimens. The dimensions of the specimen are height 90 mm and length 110 mm

Stress intensity factor at the tip of a pressurized crack

(Papamichos and Vardoulakis 1989)



- Boundary conditions at the crack

$$\Delta\pi_{xy} = 0,\ |x| < \alpha,\ y = -h$$

$$\Delta\pi_{yy} = -p,\ |x| < \alpha,\ y = -h$$

- A way of estimating stress intensity factors derives from the strain energy associated with the crack
- The strain energy for one-half crack

$$W = -\tfrac{1}{2}p\int_0^\alpha \hat{u}_y(x)\mathrm{d}x = -\tfrac{1}{2}p\sum_{j=1}^{N} b^j D_y^j$$

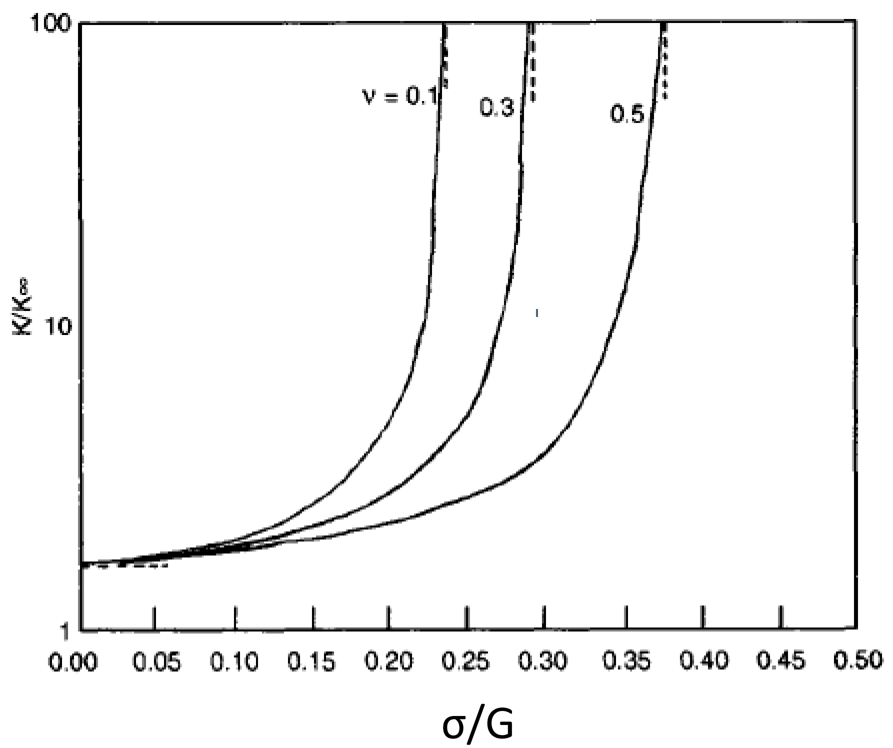
- Linear elastic fracture mechanics solutions which do not usually take into account the initial stress field give a standard relationship between the strain energy rate *∂W/∂α* and the stress intensity factor *K*
- Repeat calculation by including initial stress field

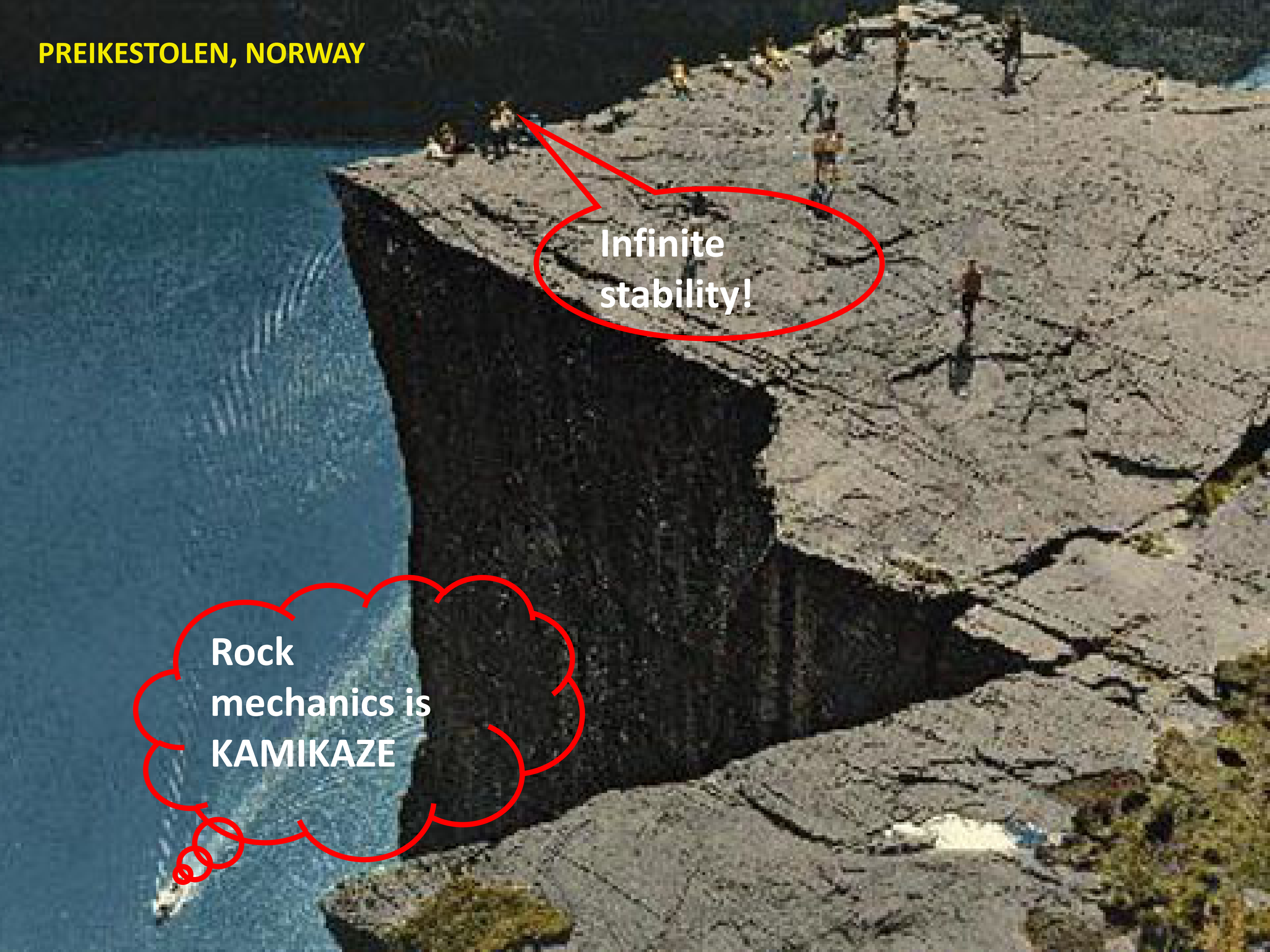
$$\frac{\partial W}{\partial \alpha} = \frac{\pi(q_1 - q_2)/(\delta_1 - \delta_2)}{2G} K^2$$

- By compute numerically *W* for slightly different crack lengths and solve for K
- K combines K_I and K_{II} modes: $K^2 = K_I^2 + K_{II}^2$

Results for K/K_∞ for $h/\alpha = 1$

- $K_\infty = p\sqrt{\alpha}$ Crack in an infinite medium
- For $\sigma = 0$: $K/K_\infty = 1.4968$
- Erdogan et al (1973) solution

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Rock mechanics is KAMIKAZE


Engineer!!
Life is risk management
Kjerag-bolten