ALERT Doctoral School 2019, Aussois, 3-4 October 2019
“The legacy of Ioannis Vardoulakis to Geomechanics”

Hydro-mechanics of porous and granular materials

Poroelasticity and beyond

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3 - 4 October 2019

SimTech

SFB 1313

University of Stuttgart

roadmap

hydro-mechanical coupling - pore scale

* from images to simulations
* Digital rock Physics
* single phase flow through porous media

quasi-static poro-elasticity

* the role of heterogeneities
* theory: state variables
* theory: constitutive equations

dynamic poro-elasticity / waves

* low- and high frequency regime
* dispersion relations

heterogeneities - fractures

* effective properties - attenuation

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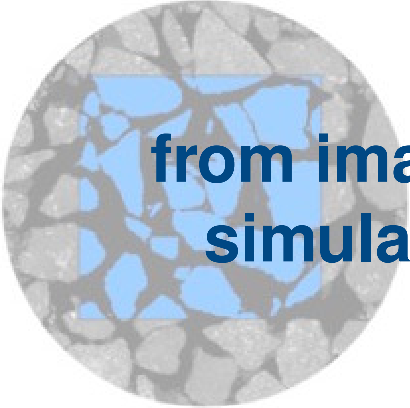
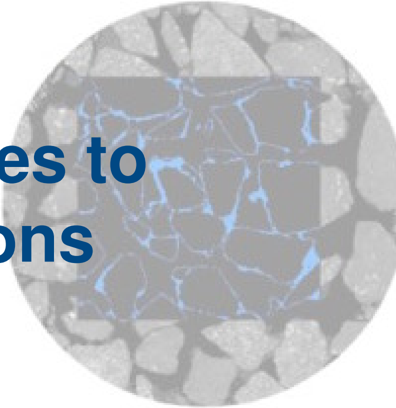
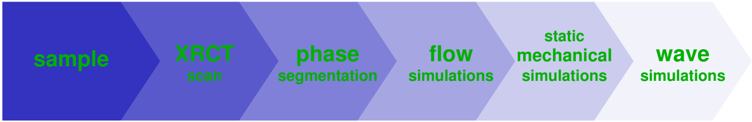
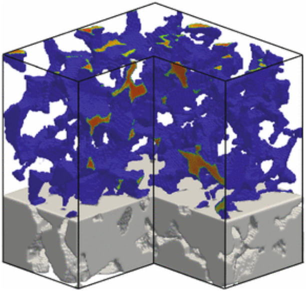



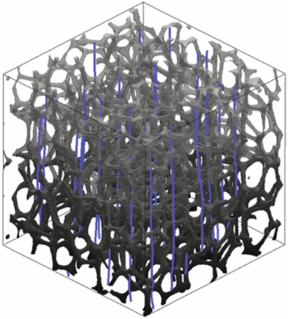
from images to simulations

Effective properties - permeability



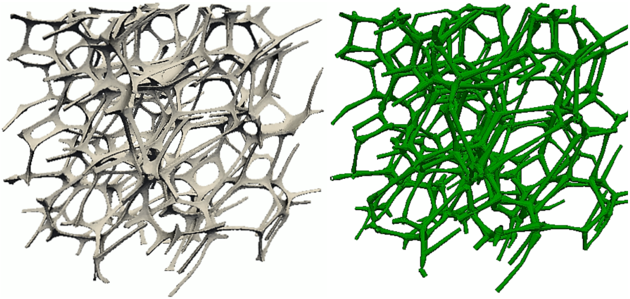
flow simulations – Stokes solver (FD)
low porous materials (sedimentary rocks)



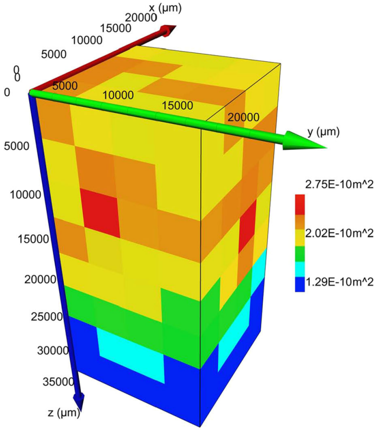


Effective properties - permeability

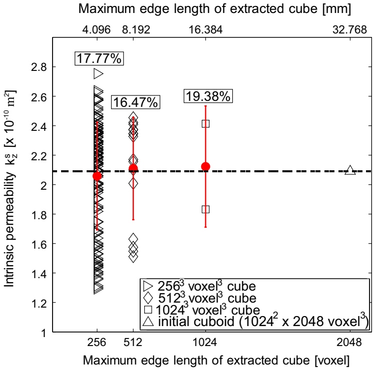
flow simulations – (Navier) Stokes solver (FD, SPH)
high porous materials (reticulite, PU foams)



Effective properties - permeability



(b)



(b)

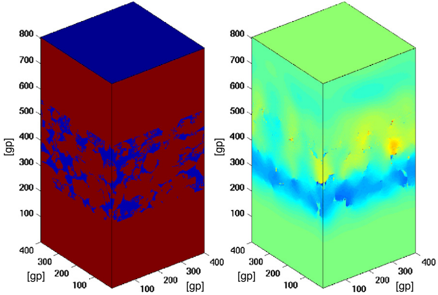
Effective properties - (dispersive) waves

waves on pore scale:

- numerical FD approach
- X-Ray CT-based

we observe:

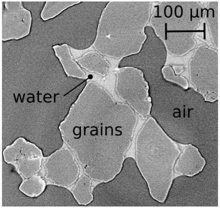
- low/high frequency waves
- determination of freq-dependent tortuosity
- higher-order wave modes e.g. slow P-wave, slow S-wave, Krauklis wave (Stoneley-guided waves)
- realistic attenuation Q(f)

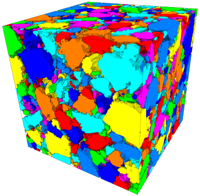


cooperation with Erik Saenger, Bochum
calculation domain 400 $\times$ 400 $\times$ 800 voxels
Saenger et al., *J. Appl. Geophys.*, **74** 2011
Saenger et al., *Geophys. Prospect.*, **64**, 2016

Saenger et al., *Solid Earth*, **7**, 2016

numerical techniques for image-based analysis





e.g. micro-fluidic cells

2-dim images: $>$ 1e6 pixels

e.g. X-ray μ-CT scans

3-dim data: $>$ 1e9 voxels

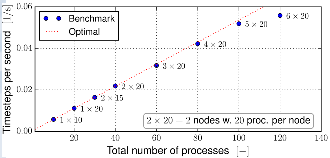
- **heterogeneous** (complex) micro- or pore-structure
- meshing is time-consuming and **almost impossible**

consequence:

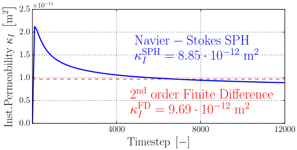
fast **explicit**, **parallel** and **meshless** solvers!

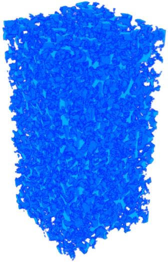
towards image-based large scale computations

Weak scalability study



Permeability computation





Software framework

- HOOMD-Blue (Highly Optimized Object-oriented Many-Particle Dynamics C++)

  Anderson et. al., J. Comp. Phys. 227 (2008)
- Supports Multi-GPU and Multi-CPU, Spatial domain decomposition
- Multi-GPU/CPU capable **weakly-compressible** conservative SPH model implemented

Hardware framework • SimTech ASES Cluster

- 52 Execute-Nodes w. 2 x Intel® Xeon® CPU (20 Cores) @ $2,80$ GHz
- QDR-Infiniband Interconnect (40 Gbit)

- Domainsize: $290 \times 290 \times 600$ Voxel
- $50 \cdot 10^6$ SPH Particles
- Voxelsize: 7.5μm
- Fontainebleau sandstone, Andrä et al. (2013)
- Computation time: 23 hrs. on 140 Cores

Currently: Cray XC40 - Hazel Hen, HLRS Stuttgart & BinAC GPU cluster, BW HPC (NVidia Tesla K80) ($> 1000^3$ voxel)

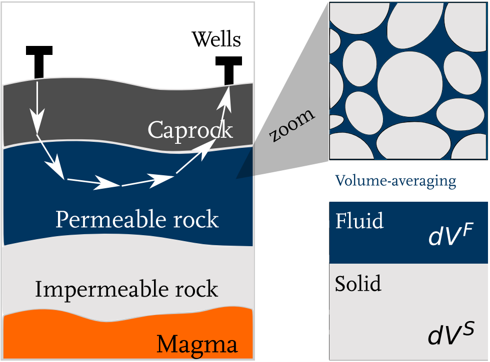
Osorno, Schirwon, Kijanski, Sivanesapillai, Steeb, Göddeke (2019), Comput. Phys. Commun. (subm.)

single-phase flow through porous media

from low- to high-Re flow

upscaling concept in porous media flow

macro-scale | pore-scale | momentum balance



$$\text{grad}\, p - \rho\, \mathbf{g} = \rho\, \mathbf{u} \cdot \rho\, \text{grad}\, \mathbf{u} - \mu\, \Delta \mathbf{u}$$

convection, diffusion

$$\langle \text{grad}\, p - \rho\, \mathbf{g} \rangle = f(\mathbf{q})$$

specific drag

effective filter laws for various flow regimes

Unidirectional (isotropic, steady) flow in $\mathbf{e}_1$-direction

$$\langle \text{grad}\, p - \rho\, \mathbf{g} \rangle_1 = -\frac{\mu^{\mathfrak{f}R}}{k^{\mathfrak{s}}} q_1$$

Darcy
Re $<$ 1

$$\langle \text{grad}\, p - \rho\, \mathbf{g} \rangle_1 = -\frac{\mu^{\mathfrak{f}R}}{k^{\mathfrak{s}}} q_1 - \frac{\rho\, c^F}{\sqrt{k^{\mathfrak{s}}}} q_1^2$$

Forchheimer
1 $<$ Re

$$\langle \text{grad}\, p - \rho\, \mathbf{g} \rangle_1 = -\frac{\rho\, c^F}{\sqrt{k^{\mathfrak{s}}}} q_1^2$$

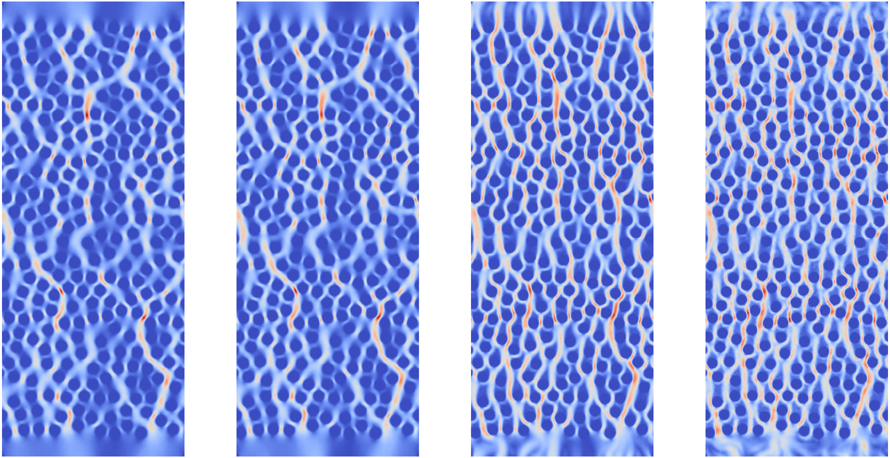
strong-inertia
$10^3 <$ Re

$$\langle \text{grad}\, p - \rho\, \mathbf{g} \rangle_1 = -\frac{\mu^{\mathfrak{f}R}}{k^{\mathfrak{s}}} q_1 - \frac{\rho^2\, \zeta}{\mu} q_1^3$$

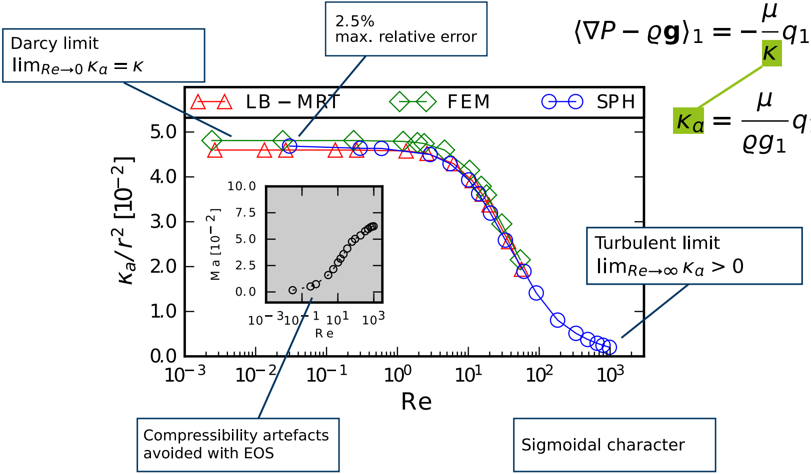
weak-inertia
1 $<$ Re $<$ 10

spatial distribution of kinetic energy - Re

Re $\mathcal{O}(10^0)$ Re $\mathcal{O}(10^1)$ Re $\mathcal{O}(10^2)$ Re $\mathcal{O}(10^3)$



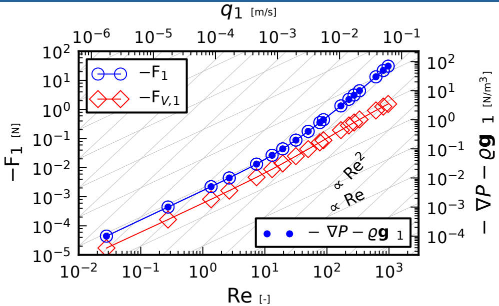
apparent hydraulic permeability



$$\langle \nabla P - \varrho \mathbf{g} \rangle_1 = -\frac{\mu}{\kappa} q_1$$

$$\kappa_a = \frac{\mu}{\varrho g_1} q_1$$

inertial transition on macro- and microscale

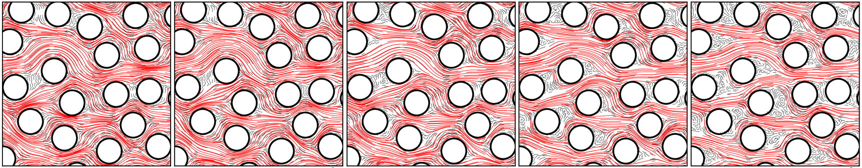


macro-scale

grad $p \propto \mathbf{q}$ for Re < 5
grad $p \propto \mathbf{q}^2$ for Re > 800
low Re: viscous effects dominate
high Re: inertia effects dominate

micro-scale

flow seperation
streamline rectification
formation of wake eddies

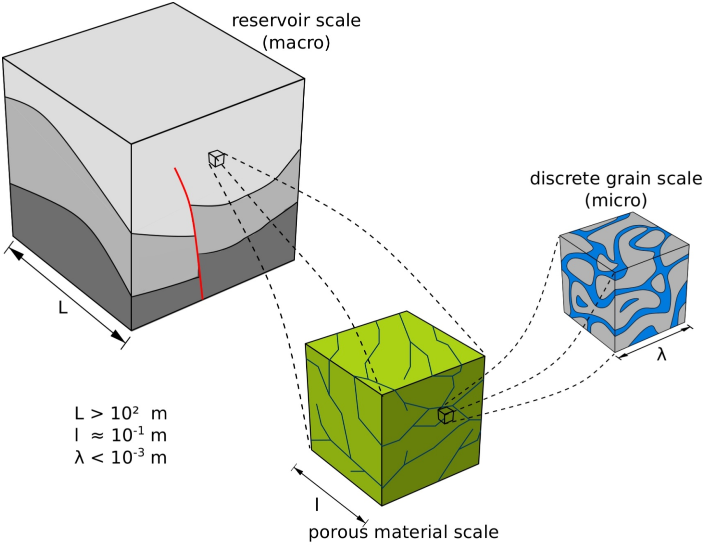


$\mathrm{Re} = \mathcal{O}(10-1)$ $\mathrm{Re} = \mathcal{O}(10^0)$ $\mathrm{Re} = \mathcal{O}(10^1)$ $\mathrm{Re} = \mathcal{O}(5 \cdot 10^1)$ $\mathrm{Re} = \mathcal{O}(10^2)$

linear poro-elasticity

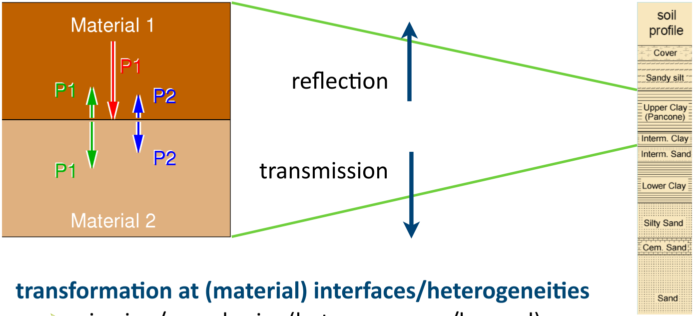
the quasi-static case

effective properties – the role of scales



$L > 10^2$ m
$l \approx 10^{-1}$ m
$\lambda < 10^{-3}$ m

continuum scale – layered “poroelastic” media



Maurice A. Biot
[1905 - 1985]

transformation at (material) interfaces/heterogeneities

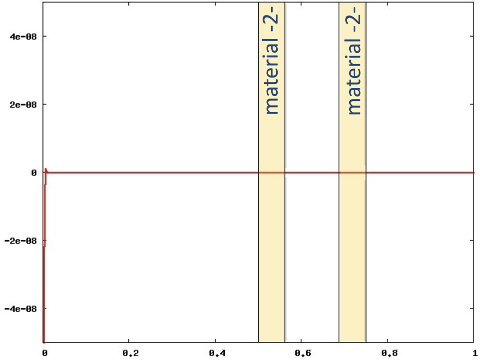
- seismics / geophysics (heterogeneous/layered)
- biomechanics (“bones“)
- materials sciences (“composites“)

→ **viscous attenuation**

Frenkel [1944], Gassmann [1951], Biot [1956],...

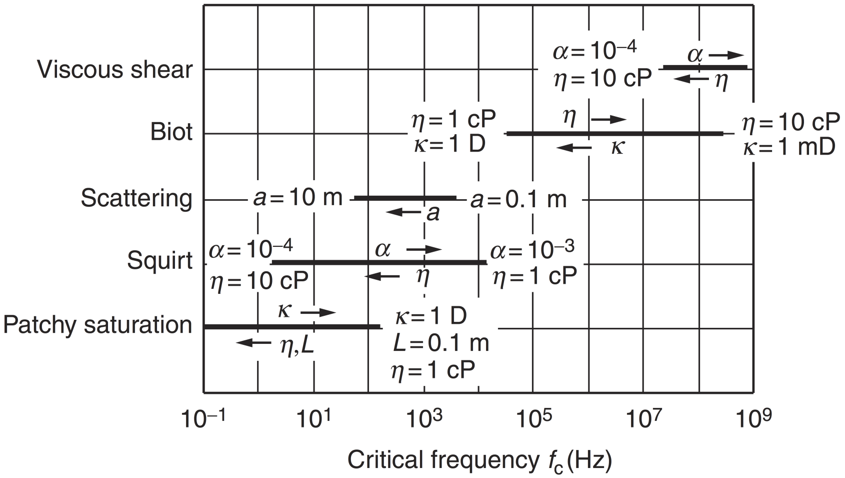
Yakov I. Frenkel
[1894 - 1952]

Biot waves in layered “poroelastic” media



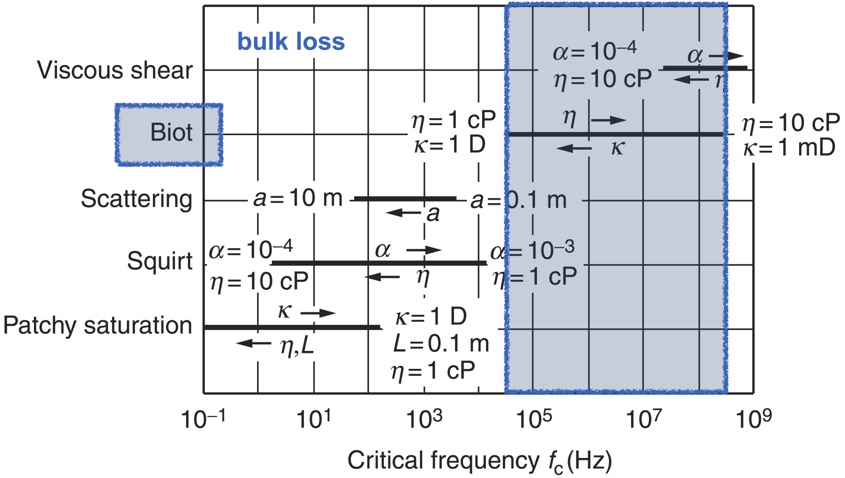
theoretical prediction: Biot [1956],
experimental validation: Plona [1980]

attenuation in porous rocks



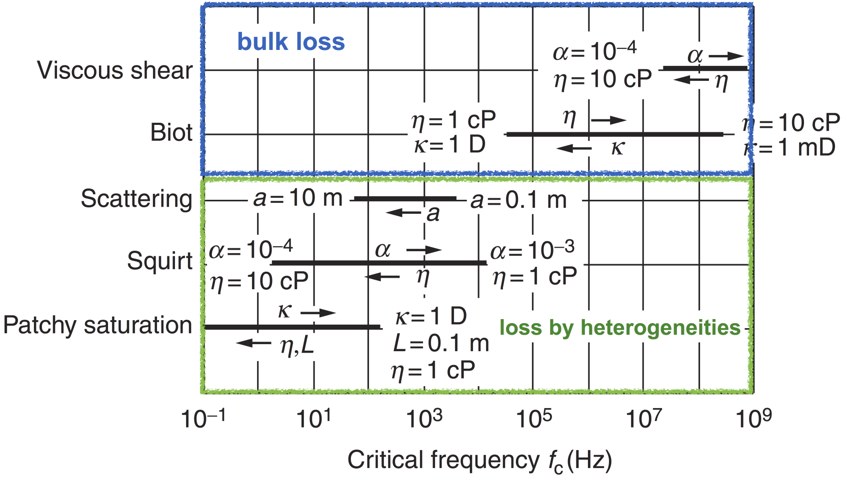
(Mavko, Mukerji & Dvorkin: The Rock Physics Handbook, 2009)

attenuation in porous rocks



(Mavko, Mukerji & Dvorkin: The Rock Physics Handbook, 2009)

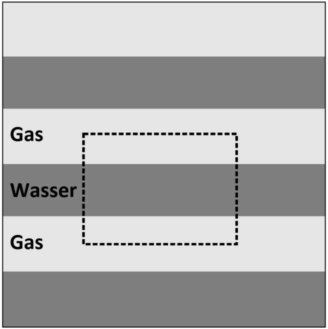
attenuation in porous rocks

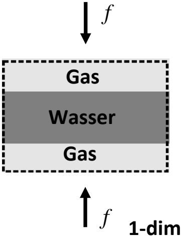


(Mavko, Mukerji & Dvorkin: The Rock Physics Handbook, 2009, modified)

"patchy" / heterogeneous saturation

reservoir rock/sandstone, **partially saturated** with water and gas
capillary effects (surface tension) are neglected





REV
Representative
Elementary **V**olume

1-dim:
White, *Geophysics*, **40**, 1975

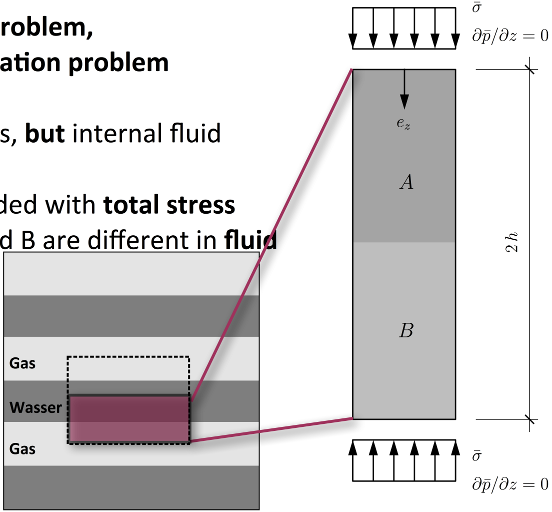
heterogeneous problem, undrained boundaries

1-dim layered problem,
„local“ consolidation problem

a) undrained BCs, **but** internal fluid redistribution

b) sample is loaded with **total stress**

c) domains A and B are different in **fluid properties**



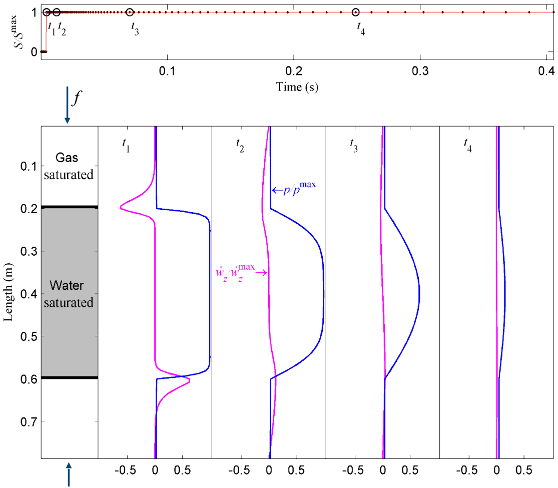
1-dim - quasi-static creep test - time domain

no fluid leak-of, only **internal redistribution**

Biot's poroelasticity

$\mathcal{P} = \{\mathbf{u}_\mathfrak{s},\, p\}$

cf. Quintal et al.
J. Geophys. Res., **116**, 2011
Geophysics, **77**, 2012



1-dim - quasi-static creep test - freq. domain

intrinsic attenuation:

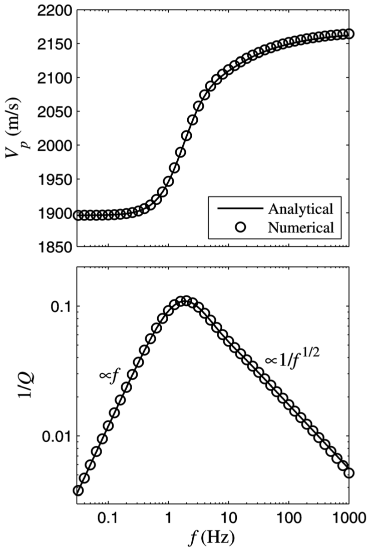
$$Q(\omega) = \frac{\mathrm{Re}\{H(\omega)\}}{\mathrm{Im}\{H(\omega)\}}$$

(P-wave) phase velocity

$$V_p(\omega) = \left(\mathrm{Re}\left\{\frac{1}{V(\omega)}\right\}\right)^{-1}$$

$$V(\omega) = \sqrt{\frac{H(\omega)}{\rho}}$$

cf. Quintal et al.
J. Geophys. Res., **116**, 2011
Geophysics, **77**, 2012



(formal) numerical homogenization approach

micro-scale: **poroelastic** medium vs. macro-scale: **viscoelastic** medium

extended Hill-Mandel condition

$$\boldsymbol{\sigma}_M : \mathbf{D}_M = \left\langle \boldsymbol{\sigma}^s_m : \mathbf{D}^m_s + \boldsymbol{\sigma}^f_m : \mathbf{D}^m_f - \hat{\mathrm{p}}^f_m \cdot \mathbf{w}^m_f \right\rangle$$

macro-scale   micro-scale

boundary conditions for micro-scale problem

$$\boldsymbol{\sigma}_M : \mathbf{D}_M = \frac{1}{V_m} \int_{\partial\Omega_m} \mathbf{v}^m_s \cdot \mathbf{t}_m \, \mathrm{d}a - \frac{1}{V_m} \int_{\partial\Omega_m} \phi \, p_m \, q^m \, \mathrm{d}a$$

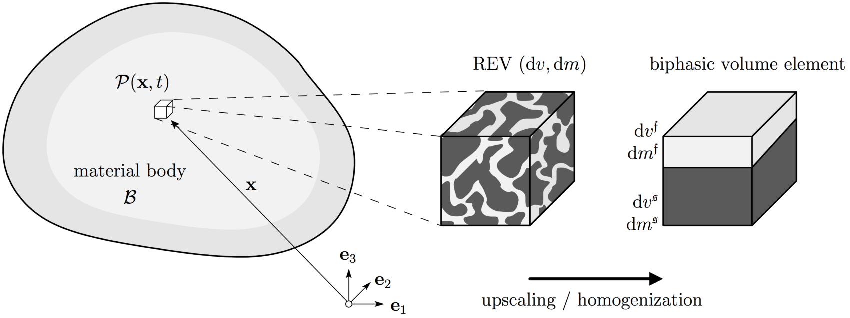
total fluxes   outflux of fluid

Jänicke et al., *Comput. Method. Appl. Mech. Engrg.*, **298**, 2016

linear poro-elasticity

some remarks to the theory

poro-elasticity - a continuum approach



poro-elasticity - assumptions & notation

mass and volume elements

$$\mathrm{d}m = \mathrm{d}m^{\mathfrak{s}} + \mathrm{d}m^{\mathfrak{f}} \qquad \text{and} \qquad \mathrm{d}v = \mathrm{d}v^{\mathfrak{s}} + \mathrm{d}v^{\mathfrak{f}}$$

poro-elasticity - assumptions & notation

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and at time t_0

$$\mathrm{d}v(\mathbf{x}, t_0) =: \mathrm{d}v_0, \quad \mathrm{d}v^{\mathfrak{s}}(\mathbf{x}, t_0) =: \mathrm{d}v_0^{\mathfrak{s}} \quad \text{and} \quad \mathrm{d}v^{\mathfrak{f}}(\mathbf{x}, t_0) =: \mathrm{d}v_0^{\mathfrak{f}}$$

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volume fractions $n^\alpha := \mathrm{d}v^\alpha / \mathrm{d}v$ and Eulerian porosity $n^{\mathfrak{f}}(\mathbf{x}, t)$

$$n^{\mathfrak{f}} := \frac{\mathrm{d}v^{\mathfrak{f}}}{\mathrm{d}v} = \frac{\mathrm{d}v - \mathrm{d}v^{\mathfrak{s}}}{\mathrm{d}v}$$

poro-elasticity - assumptions & notation

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note: it's a **(non-linear)** Eulerian field variable

poro-elasticity - assumptions & notation

mass and volume elements

$$\mathrm{d}m = \mathrm{d}m^{\mathfrak{s}} + \mathrm{d}m^{\mathfrak{f}} \qquad \text{and} \qquad \mathrm{d}v = \mathrm{d}v^{\mathfrak{s}} + \mathrm{d}v^{\mathfrak{f}}$$

and at time t_0

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volume fractions $n^\alpha := \mathrm{d}v^\alpha / \mathrm{d}v$ and Eulerian porosity $n^{\mathfrak{f}}(\mathbf{x}, t)$

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note: it's a **(non-linear)** Eulerian field variable
(linear) Lagrangian porosity $\phi(\mathbf{x}, t)$

$$\phi := \frac{\mathrm{d}v^{\mathfrak{f}}}{\mathrm{d}v_0} = \frac{\mathrm{d}v - \mathrm{d}v^{\mathfrak{s}}}{\mathrm{d}v_0} \quad \text{with} \quad \phi_0 = \frac{\mathrm{d}v_0^{\mathfrak{f}}}{\mathrm{d}v_0}.$$

poro-elasticity - assumptions & notation

note: in linear poro-elasticity

$$\mathrm{lin}(n^{\mathfrak{f}}) = \phi \qquad \text{(proof yourself!)}$$

poro-elasticity - assumptions & notation

note: in linear poro-elasticity

$$\mathrm{lin}(n^{\mathfrak{f}}) = \phi \qquad \text{(proof yourself!)}$$

effective and partial densities

$$\rho^{\mathfrak{s}R} := \frac{\mathrm{d}m^{\mathfrak{s}}}{\mathrm{d}v^{\mathfrak{s}}} \quad \text{and} \quad \rho^{\mathfrak{f}R} := \frac{\mathrm{d}m^{\mathfrak{f}}}{\mathrm{d}v^{\mathfrak{f}}},$$

$$\rho^{\mathfrak{s}} := \frac{\mathrm{d}m^{\mathfrak{s}}}{\mathrm{d}v} \quad \text{and} \quad \rho^{\mathfrak{f}} := \frac{\mathrm{d}m^{\mathfrak{f}}}{\mathrm{d}v},$$

$$\rho := \frac{\mathrm{d}m}{\mathrm{d}v}.$$

poro-elasticity - assumptions & notation

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$$\rho := \frac{\mathrm{d}m}{\mathrm{d}v}.$$

Remark: a saturation condition is fullfilled with

$$\sum_{\alpha} n^{\alpha} \equiv 1$$

poro-elasticity - kinematics

displacement of the skeleton $\mathbf{u}_\mathfrak{s}$ and time derivatives

$$\ddot{\mathbf{u}}_\mathfrak{s} = \dot{\mathbf{v}}_\mathfrak{s} = \mathbf{a}_\mathfrak{s} \qquad \text{and} \qquad \ddot{\mathbf{u}}_\mathfrak{f} = \dot{\mathbf{v}}_\mathfrak{f} = \mathbf{a}_\mathfrak{f}.$$

poro-elasticity - kinematics

displacement of the skeleton $\mathbf{u}_\mathfrak{s}$ and time derivatives

$$\ddot{\mathbf{u}}_\mathfrak{s} = \dot{\mathbf{v}}_\mathfrak{s} = \mathbf{a}_\mathfrak{s} \qquad \text{and} \qquad \ddot{\mathbf{u}}_\mathfrak{f} = \dot{\mathbf{v}}_\mathfrak{f} = \mathbf{a}_\mathfrak{f}.$$

seepage (or simple relative) velocity & Darcy (or filter) velocity

$$\mathbf{w}_\mathfrak{f} = \mathbf{v}_\mathfrak{f} - \mathbf{v}_\mathfrak{s} \qquad \& \qquad \mathbf{q}_\mathfrak{f} = (\phi_0)\,(\mathbf{v}_\mathfrak{f} - \mathbf{v}_\mathfrak{s})$$

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solid strain

$$\begin{aligned} \boldsymbol{\varepsilon}_\mathfrak{s} &= \frac{1}{2}\left(\operatorname{grad}\mathbf{u}_\mathfrak{s} + \operatorname{grad}^T\mathbf{u}_\mathfrak{s}\right) \\ &= \operatorname{dev}(\boldsymbol{\varepsilon}_\mathfrak{s}) + \operatorname{vol}(\boldsymbol{\varepsilon}_\mathfrak{s}) =: \boldsymbol{\gamma}_\mathfrak{s} + e_\mathfrak{s}\,\mathbf{I} \end{aligned}$$

poro-elasticity - kinematics

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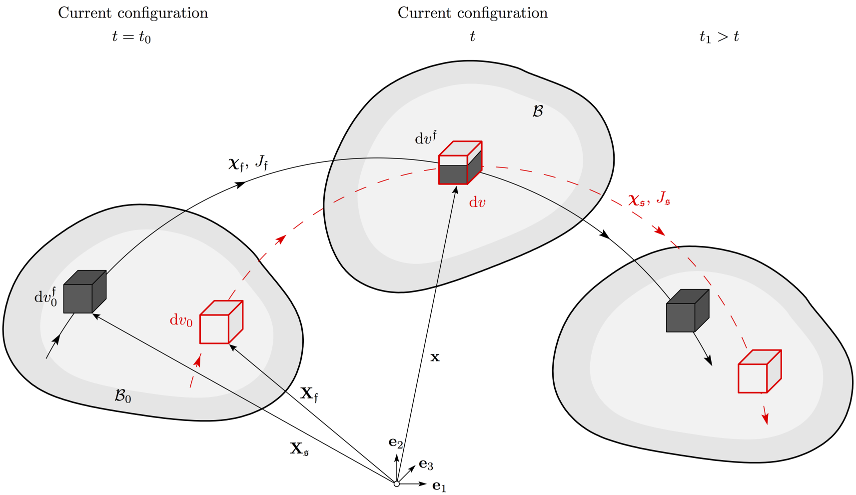
Remark: In continua, the so-called material time derivative of a vectorial field variable $\boldsymbol{\Psi}^\alpha$ in a mixture is given by $(\boldsymbol{\Psi}^\alpha)'_\alpha = \partial_t\boldsymbol{\Psi} + \operatorname{grad}\boldsymbol{\Psi}\cdot\mathbf{v}_\alpha$. The first term is denoted as (linear) local or partial time derivative while the 2nd term is a convective (non-linear) term which vanishes in linear models (like in linear poro-elasticity).

poro-elasticity - kinematics



poro-elasticity - kinematics

alternative interpretation of volumetrical deformation - **volume map**

$$e_{\mathfrak{s}} = \frac{\mathrm{d}v - \mathrm{d}v_0}{\mathrm{d}v_0} \qquad \text{and} \qquad e_{\mathfrak{f}} = \frac{\mathrm{d}v^{\mathfrak{f}} - \mathrm{d}v_0^{\mathfrak{f}}}{\mathrm{d}v_0^{\mathfrak{f}}}.$$

poro-elasticity - kinematics

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why? Linearized map of volume elements (solid skeleton)

$$\mathrm{d}v = J_{\mathfrak{s}}\,\mathrm{d}v_0 \quad \text{with} \quad \det \mathbf{F}_{\mathfrak{s}} =: J_{\mathfrak{s}} \quad \text{and} \quad \mathrm{lin}(J_{\mathfrak{s}}) = e_{\mathfrak{s}} + 1 = \mathrm{div}\,\mathbf{u}_{\mathfrak{s}} + 1$$

poro-elasticity - kinematics

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macro-scopical volume change / could be measured globally

poro-elasticity - kinematics

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macro-scopical volume change / could be measured globally

linearized map of volume elements (fluid)

$$\mathrm{d}v^{\mathfrak{f}} = J_{\mathfrak{f}}\,\mathrm{d}v_0^{\mathfrak{f}} \quad \text{with} \quad \det \mathbf{F}_{\mathfrak{f}} =: J_{\mathfrak{f}} \quad \text{and} \quad \mathrm{lin}(J_{\mathfrak{f}}) = e_{\mathfrak{f}} + 1 = \mathrm{div}\,\mathbf{u}_{\mathfrak{f}} + 1$$

poro-elasticity - kinematics

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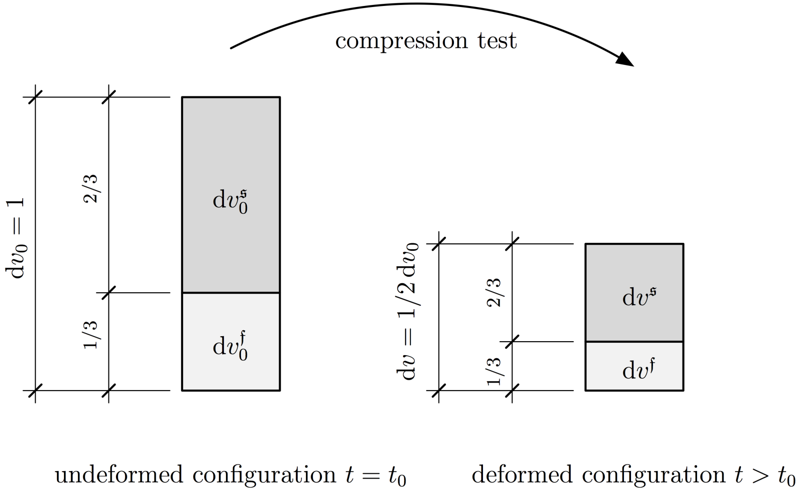
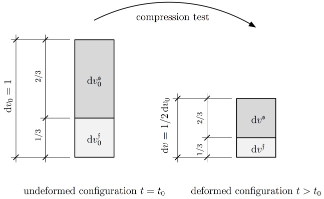
micro-scopical volume change / could be measured only on pore-scale

poro-elasticity - kinematics

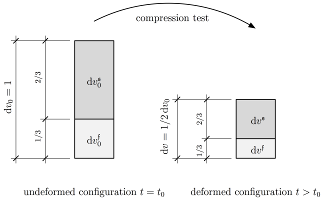


simple "Gedankenexperiment" of a hydrostatic test (isotropic, homogeneous,. . .)

poro-elasticity - kinematics



poro-elasticity - kinematics



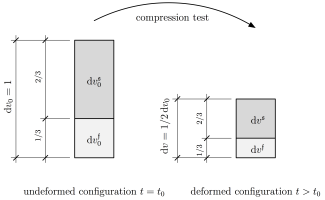
volumetric deformation of solid skeleton and fluid phase

$$e_{\mathfrak{s}} = \frac{dv - dv_0}{dv_0} = \frac{1/2\,dv_0 - dv_0}{dv_0} = -\frac{1}{2},$$

and

$$e_{\mathfrak{f}} = \frac{dv^{\mathfrak{f}} - dv_0^{\mathfrak{f}}}{dv_0^{\mathfrak{f}}} = \frac{1/3\,dv - 1/3\,dv_0}{1/3\,dv_0} = \frac{dv - dv_0}{dv_0} = -\frac{1}{2}$$

poro-elasticity - kinematics



Remark: In continuum mixture theory often (e.g. de Boer, 2005) a transport theorem is defined

$$\mathrm{d}v = J_\alpha\, \mathrm{d}v_0^\alpha \qquad \rightsquigarrow \qquad \bar{e}_\alpha = \frac{\mathrm{d}v - \mathrm{d}v_0^\alpha}{\mathrm{d}v_0^\alpha}$$

Note that this transport theorem neither has a simple geometric interpretations nor it is consistent with the mapping rules of measurable kinematic quantities (and is not consistent with the simple experiment)

poro-elasticity - kinematics

increment of fluid content (Biot & Willis, 1957)

$$\zeta = \phi_0 \left(e_{\mathfrak{s}} - e_{\mathfrak{f}} \right).$$

poro-elasticity - kinematics

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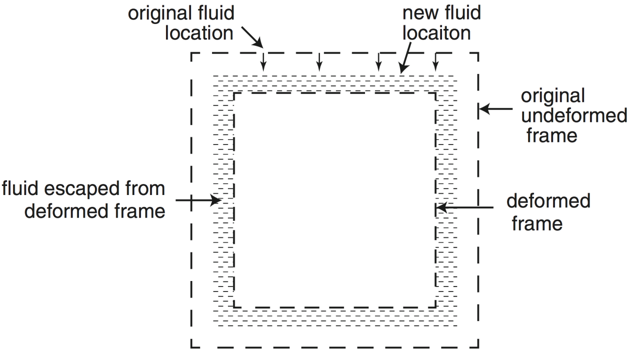

hydrostatic experiment - volume change of fluid/solid; from Cheng (2016), Poroelasticity

poro-elasticity - kinematics

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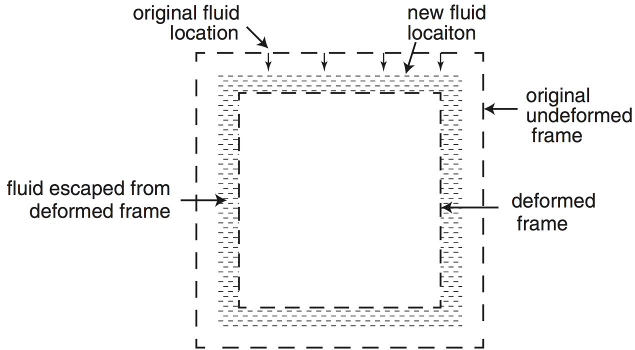

hydrostatic experiment - volume change of fluid/solid; from Cheng (2016), Poroelasticity

Remark: Other definitions are around (cf. Wang, 2000; Rice & Cleary, 1976)

poro-elasticity - balance equations

balance of mass of the single constituents:

$$\mathcal{M}^{\mathfrak{s}} = \int_{\mathcal{B}} \rho^{\mathfrak{s}} \, \mathrm{d}v = \mathcal{M}^{\mathfrak{s}}_0 = \text{const.} \quad \text{and} \quad \mathcal{M}^{\mathfrak{f}} = \int_{\mathcal{B}} \rho^{\mathfrak{f}} \, \mathrm{d}v = \mathcal{M}^{\mathfrak{f}}_0 = \text{const.}.$$

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or in local form (after “some” standard algebra)

$$\partial_t (n^{\mathfrak{s}} \, \rho^{\mathfrak{s}R}) + \mathrm{div}(n^{\mathfrak{s}} \, \rho^{\mathfrak{s}R} \, \mathbf{v}_{\mathfrak{s}}) = 0,$$

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Note: These expressions are non-linear as e.g. convective terms in the time derivatives are included. Further, (non-linear) products of volume fractions, densities and velocities appear.

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linearization

we write the balance of mass of the constituent φ^α alternatively

$$\mathcal{M}^\alpha = \int_{\mathcal{B}} \rho^\alpha \, \mathrm{d}v = \int_{\mathcal{B}_0} \rho_0^\alpha \, \mathrm{d}v_0^\alpha = \mathcal{M}_0^\alpha .$$

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or with $\mathrm{d}v = J_\alpha \, \mathrm{d}v_0^\alpha$ (solid skeleton)

$$\int_{\mathcal{B}} \rho^\alpha \, J_\alpha \, \mathrm{d}v_0^\alpha = \int_{\mathcal{B}_0} \rho_0^\alpha \, \mathrm{d}v_0^\alpha ,$$

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$$\rho^\alpha \, J_\alpha = \rho_0^\alpha .$$

and with the linearized Jacobian $\mathrm{lin}(J_\alpha) = e_\alpha + 1$

$$n^\alpha \, \rho^{\alpha R} \, (e_\alpha + 1) = n_0^\alpha \, \rho_0^{\alpha R} \qquad \text{(nonlinear terms)}$$

linearization

linearization of non-linear terms around

$$\mathbf{x}_0 = \left[n^\alpha(\mathbf{x},\, t_0),\, \rho^{\alpha R}(\mathbf{x},\, t_0),\, e_\alpha(\mathbf{x},\, t_0)\right]^T = \left[n_0^\alpha,\, \rho_0^{\alpha R},\, 0\right]^T$$

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leads to (be careful: $n_0 \neq 0$ and $\rho_0^{\alpha R} \neq 0$ but $e_{\alpha,\,0} = 0$)

$$\operatorname{lin}(\mathcal{F}(n^\alpha\,\rho^{\alpha R})) \quad = \quad \mathcal{F}_0 + \left.\frac{\partial\left[(n_0^\alpha + \epsilon\,\Delta n^\alpha)(\rho_0^{\alpha R} + \epsilon\,\Delta\rho^{\alpha R})\right]}{\partial\epsilon}\right|_{\epsilon=0},$$

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&= \mathcal{F}_0 + n_0^\alpha\, \Delta\rho^{\alpha R} + \rho^{\alpha R}\, \Delta n^\alpha,
\end{aligned}$$

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&= n^\alpha\,\rho_0^{\alpha R} + n_0^\alpha\,\rho^{\alpha R} - n_0^\alpha\,\rho_0^{\alpha R}.
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and (trivial)

$$\mathrm{lin}(\mathcal{F}(n^\alpha\, \rho^{\alpha R}\, e_\alpha) = n_0^\alpha\, \rho_0^{\alpha R}\, e_\alpha,$$

linearization

and finally we get the linearized mass balances ($\phi = n^{\mathfrak{f}} = 1 - n^{\mathfrak{s}}$)

$$\phi = \phi(\rho^{\mathfrak{s}R},\, e_{\mathfrak{s}}) = 2\,\phi_0 - 1 + (1-\phi_0)\left(\frac{\rho^{\mathfrak{s}R}}{\rho_0^{\mathfrak{s}R}} + e_{\mathfrak{s}}\right)$$

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Remark: porosity is a (dependent) function of volumetric deformation of the solid and the fluid constituent and density

poro-elasticity - balance of momentum

partial stress tensor of solid skeleton (volumetric / deviatoric split)

$$\boldsymbol{\sigma}^{\mathfrak{s}} = \mathrm{vol}(\boldsymbol{\sigma}^{\mathfrak{s}}) + \mathrm{dev}(\boldsymbol{\sigma}^{\mathfrak{s}}) = \frac{1}{3}\,\mathrm{tr}(\boldsymbol{\sigma}^{\mathfrak{s}})\,\mathbf{I} + \mathrm{dev}(\boldsymbol{\sigma}^{\mathfrak{s}}) := s^{\mathfrak{s}}\,\mathbf{I} + \boldsymbol{\tau}^{\mathfrak{s}}.$$

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partial stress tensor of fluid $\boldsymbol{\sigma}^{\mathfrak{f}} = s^{\mathfrak{f}}\,\mathbf{I} = -p\,\mathbf{I}$ (no viscous shear stresses - “creeping flow” conditions are assumed in poro-elasticity)

$$\boldsymbol{\sigma}^{\mathfrak{f}} = s^{\mathfrak{f}}\,\mathbf{I} = -p\,\mathbf{I}$$

note: sign convention of pressure

poro-elasticity - balance of momentum

partial stress tensor of solid skeleton (volumetric / deviatoric split)

$$\boldsymbol{\sigma}^{\mathfrak{s}} = \mathrm{vol}(\boldsymbol{\sigma}^{\mathfrak{s}}) + \mathrm{dev}(\boldsymbol{\sigma}^{\mathfrak{s}}) = \frac{1}{3}\,\mathrm{tr}(\boldsymbol{\sigma}^{\mathfrak{s}})\,\mathbf{I} + \mathrm{dev}(\boldsymbol{\sigma}^{\mathfrak{s}}) := s^{\mathfrak{s}}\,\mathbf{I} + \boldsymbol{\tau}^{\mathfrak{s}}.$$

partial stress tensor of fluid $\boldsymbol{\sigma}^{\mathfrak{f}} = s^{\mathfrak{f}}\,\mathbf{I} = -p\,\mathbf{I}$ (no viscous shear stresses - “creeping flow” conditions are assumed in poro-elasticity)

$$\boldsymbol{\sigma}^{\mathfrak{f}} = s^{\mathfrak{f}}\,\mathbf{I} = -p\,\mathbf{I}$$

note: sign convention of pressure

total stresses (of the mixture) as sum of partial stresses

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathfrak{s}} + \boldsymbol{\sigma}^{\mathfrak{f}} = (s^{\mathfrak{s}} + s^{\mathfrak{f}})\,\mathbf{I} + \boldsymbol{\tau}^{\mathfrak{s}} =: \sigma^{M}\,\mathbf{I} + \boldsymbol{\tau}$$

poro-elasticity - balance of momentum

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with total mean stress σ^{M}

poro-elasticity - balance of momentum

global form of partial balance of momentum

$$\frac{\partial}{\partial t}(\boldsymbol{\mathcal{J}}^{\mathfrak{s}}) = \boldsymbol{\mathcal{F}}^{\mathfrak{s}}_{\mathcal{B}} + \boldsymbol{\mathcal{F}}^{\mathfrak{s}}_{\partial\mathcal{B}} + \hat{\boldsymbol{\mathcal{P}}}^{\mathfrak{s}}$$

poro-elasticity - balance of momentum

global form of partial balance of momentum

$$\frac{\partial}{\partial t}(\mathcal{J}^{\mathfrak{s}}) = \mathcal{F}^{\mathfrak{s}}_{\mathcal{B}} + \mathcal{F}^{\mathfrak{s}}_{\partial\mathcal{B}} + \hat{\mathcal{P}}^{\mathfrak{s}}$$

Axiom: momentum $\mathcal{J}^{\mathfrak{s}}$ is changed by the sum of the body forces $\mathcal{F}^{\mathfrak{s}}_{\mathcal{B}}$, contact forces $\mathcal{F}^{\mathfrak{s}}_{\partial\mathcal{B}}$, and interaction forces $\hat{\mathcal{P}}^{\mathfrak{s}} = -\hat{\mathcal{P}}^{\mathfrak{f}}$

poro-elasticity - balance of momentum

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and on local form for the solid skeleton

$$\rho^{\mathfrak{s}}\,\mathbf{a}_{\mathfrak{s}} - \operatorname{div}\boldsymbol{\sigma}^{\mathfrak{s}} = \rho^{\mathfrak{s}}\,\mathbf{b} - \hat{\mathbf{p}}^{\mathfrak{f}}$$

poro-elasticity - balance of momentum

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and on local form for the solid skeleton

$$\rho^{\mathfrak{s}}\,\mathbf{a}_{\mathfrak{s}} - \operatorname{div}\boldsymbol{\sigma}^{\mathfrak{s}} = \rho^{\mathfrak{s}}\,\mathbf{b} - \hat{\mathbf{p}}^{\mathfrak{f}}$$

and the pore fluid

$$\rho^{\mathfrak{f}}\,\mathbf{a}_{\mathfrak{f}} + \operatorname{div}(\phi\,p\,\mathbf{I}) = \rho^{\mathfrak{f}}\,\mathbf{b} + \hat{\mathbf{p}}^{\mathfrak{f}}$$

with local momentum exchange (fluid-solid) $\hat{\mathbf{p}}^{\mathfrak{f}}$

poro-elasticity - constitutive equations

thermodynamical-consistent framework

balance of entropy of the mixture (2nd law of TD - in form of CD-inequality) has to be fulfilled!

poro-elasticity - constitutive equations

results from the entropy inequality

set of process variables $\mathcal{P} = \{\boldsymbol{\gamma}_{\mathfrak{s}},\, e_{\mathfrak{s}},\, \zeta\}$ and (a most general quadratic) strain energy function $W = W(\boldsymbol{\gamma}_{\mathfrak{s}},\, e_{\mathfrak{s}},\, \zeta)$

$$\boldsymbol{\tau} = \frac{\partial W}{\partial \boldsymbol{\gamma}_{\mathfrak{s}}}, \qquad \sigma^M = \frac{\partial W}{\partial e_{\mathfrak{s}}}, \qquad p = \frac{\partial W}{\partial \zeta}.$$

poro-elasticity - constitutive equations

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$$\begin{aligned} W &= \mathrm{dev}(W) + \mathrm{vol}(W) = W(\boldsymbol{\gamma}_\mathfrak{s}) + W(e_\mathfrak{s}) + W(e_\mathfrak{s},\, \zeta) + W(\zeta), \\ &= a\,[\boldsymbol{\gamma}_\mathfrak{s} : \boldsymbol{\gamma}_\mathfrak{s}] + b\, e_\mathfrak{s}^2 + c\, e_\mathfrak{s}\, \zeta + d\, \zeta^2, \end{aligned}$$

poro-elasticity - constitutive equations

results from the entropy inequality

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we observe that the elastic part of poro-elasticity is comprising four elastic parameters $\mathcal{M} = (a,\, b,\, c,\, d)$. How are these parameters related to "physical" quantities like

$$\bar{\mathcal{M}} = (G,\, K_u,\, \alpha,\, M) \qquad \text{or} \qquad \tilde{\mathcal{M}} = (G,\, K,\, K^\mathfrak{s},\, K^\mathfrak{f})$$

poro-elasticity - constitutive equations

strain energy function & results from the CD-inequality

poro-elasticity - constitutive equations

strain energy function & results from the CD-inequality

deviatoric ("shear") part

$$\boldsymbol{\tau} = \frac{\partial W}{\partial \boldsymbol{\gamma}_{\mathfrak{s}}} = 2\, a\, \boldsymbol{\gamma}_{\mathfrak{s}}$$

poro-elasticity - constitutive equations

strain energy function & results from the CD-inequality

deviatoric (“shear”) part

$$\boldsymbol{\tau} = \frac{\partial W}{\partial \boldsymbol{\gamma}_{\mathfrak{s}}} = 2\, a\, \boldsymbol{\gamma}_{\mathfrak{s}} = 2\, G\, \boldsymbol{\gamma}_{\mathfrak{s}} \quad \text{cf. Hooke's law}$$

poro-elasticity - constitutive equations

strain energy function & results from the CD-inequality

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$$\boldsymbol{\tau} = \frac{\partial W}{\partial \boldsymbol{\gamma}_{\mathfrak{s}}} = 2\,a\,\boldsymbol{\gamma}_{\mathfrak{s}} = 2\,G\,\boldsymbol{\gamma}_{\mathfrak{s}} \quad \text{cf. Hooke's law}$$

we re-write the volumetric part, i.e. the scalar stress-strain relations in matrix form

$$\begin{bmatrix} \sigma^M \\ p \end{bmatrix} = \begin{bmatrix} 2\,b & c \\ c & 2\,d \end{bmatrix} \begin{bmatrix} e_{\mathfrak{s}} \\ \zeta \end{bmatrix}$$

poro-elasticity - constitutive equations

strain energy function & results from the CD-inequality

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$$\begin{bmatrix} \sigma^M \\ p \end{bmatrix} = \begin{bmatrix} 2\,b & c \\ c & 2\,d \end{bmatrix} \begin{bmatrix} e_{\mathfrak{s}} \\ \zeta \end{bmatrix}$$

or alternatively

$$\begin{bmatrix} e_{\mathfrak{s}} \\ \zeta \end{bmatrix} = \frac{1}{4\,b\,d - c^2} \begin{bmatrix} 2\,d & -c \\ -c & 2\,b \end{bmatrix} \begin{bmatrix} \sigma^M \\ p \end{bmatrix}$$

poro-elasticity - constitutive equations

strain energy function & results from the CD-inequality

deviatoric (“shear”) part

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interpretation of $(b,\ c,\ d)$ remains . . .

poro-elasticity - constitutive equations

“physical meaning” of (b, c, d) derived by “Gedankenexperimente”

poro-elasticity - constitutive equations

“physical meaning” of $(b,\ c,\ d)$ derived by “Gedankenexperimente”

undrained bulk modulus (K_u: undrained conditions, $\zeta = \mathrm{const.}$)

$$\left.\frac{\partial \sigma^M}{\partial e_{\mathfrak{s}}}\right|_{\zeta} = 2\,b =: K_\mathrm{u}.$$

poro-elasticity - constitutive equations

"physical meaning" of $(b,\ c,\ d)$ derived by "Gedankenexperimente"

undrained bulk modulus (K_u: undrained conditions, $\zeta = \mathrm{const.}$)

$$\left.\frac{\partial \sigma^M}{\partial e_{\mathfrak{s}}}\right|_{\zeta} = 2\,b =: K_\mathrm{u}.$$

Skempton parameter (B: undrained conditions, $\zeta = \mathrm{const.}$)

$$0 = -c\,\sigma^M + 2\,b\,p \quad \Longleftrightarrow \quad \left.\frac{\partial p}{\partial \sigma^M}\right|_{\zeta} = \frac{c}{2\,b} =: B$$

poro-elasticity - constitutive equations

"physical meaning" of $(b,\ c,\ d)$ derived by "Gedankenexperimente"

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drained bulk modulus (K: drained conditions $p = \mathrm{const.}$)

$$\left.\frac{\partial e_{\mathfrak{s}}}{\partial \sigma^M}\right|_{p} = \frac{2\,d}{4\,b\,d - c^2} =: \frac{1}{K}$$

poro-elasticity - constitutive equations

Biot-Willis coefficient (α: drained conditions, $p = \mathrm{const.}$)

$$\left.\frac{\partial \zeta}{\partial e_{\mathfrak{s}}}\right|_{p} = -\frac{c}{2d} =: \alpha$$

poro-elasticity - constitutive equations

Biot-Willis coefficient (α: drained conditions, $p = \mathrm{const.}$)

$$\left.\frac{\partial \zeta}{\partial e_{\mathfrak{s}}}\right|_{p} = -\frac{c}{2d} =: \alpha$$

specific storage capacity (s variation in fluid increment per change in pore pressure, i.e. "storage of fluid volume"). Two conditions:

a) $e_{\mathfrak{s}} = \mathrm{const.}$ (skeleton is not volumetrically deformed)

$$\left.\frac{\partial \zeta}{\partial p}\right|_{e_{\mathfrak{s}}} = \frac{1}{2\,d} =: s_{e_{\mathfrak{s}}} =: \frac{1}{M}$$

poro-elasticity - constitutive equations

Biot-Willis coefficient (α: drained conditions, $p = \text{const.}$)

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$$\left.\frac{\partial \zeta}{\partial p}\right|_{e_{\mathfrak{s}}} = \frac{1}{2\,d} =: s_{e_{\mathfrak{s}}} =: \frac{1}{M}$$

b) $\sigma^M = \text{const.}$ (mean stress does not change)

$$\left.\frac{\partial \zeta}{\partial p}\right|_{\sigma^M} = \frac{2b}{4\,b\,d - c^2} =: s_{\sigma^M}$$

poro-elasticity - constitutive equations

1st conclusion: set of material parameters $\{G,\, K_\mathrm{u},\, \alpha,\, M\}$

$$\begin{bmatrix} \sigma^M \\ p \end{bmatrix} = \begin{bmatrix} K_\mathrm{u} & -\alpha\, M \\ -\alpha\, M & M \end{bmatrix} \begin{bmatrix} e_\mathfrak{s} \\ \zeta \end{bmatrix}$$

poro-elasticity - constitutive equations

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or inverse

$$\begin{bmatrix} e_\mathfrak{s} \\ \zeta \end{bmatrix} = \frac{1}{K_\mathrm{u} - \alpha^2\, M} \begin{bmatrix} 1 & \alpha \\ \alpha & K_\mathrm{u}/M \end{bmatrix} \begin{bmatrix} \sigma^M \\ p \end{bmatrix}$$

poro-elasticity - constitutive equations

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Remark: The set of (three) parameters $\{K_\mathrm{u},\, \alpha,\, M\}$ can be replaced by any combination of the introduced quantities like

$$K = K_\mathrm{u} - \alpha^2\, M, \qquad \frac{s_{e_\mathfrak{s}}}{s_{\sigma^M}} = \frac{M}{K_\mathrm{u}}, \qquad B = \frac{\alpha\, M}{K_\mathrm{u}}$$

or (often used) $\{K,\, K^\mathfrak{f},\, K^\mathfrak{s}\}$

poro-elasticity - the role of porosity

alternative choice of state variables

re-starting from the constitutive relation of the pore fluid

$$p = K^{\mathfrak{f}} \left[\frac{\rho^{\mathfrak{f}R}}{\rho_0^{\mathfrak{f}R}} - 1 \right] \quad \text{or} \quad \rho^{\mathfrak{f}R} = \rho_0^{\mathfrak{f}R} \left[\frac{p}{K^{\mathfrak{f}}} + 1 \right]$$

poro-elasticity - the role of porosity

alternative choice of state variables

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the expression $\rho^{\mathfrak{f}R}(p)$ can be used in the balance of mass to replace effective density with pore pressure which leads to

$$\phi = \phi_0 - \phi_0 \left[\frac{p}{K^{\mathfrak{f}}} + e_{\mathfrak{f}} \right]$$

poro-elasticity - the role of porosity

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$$\phi = \phi_0 - \phi_0 \left[\frac{p}{K^{\mathfrak{f}}} + e_{\mathfrak{f}} \right]$$

using the relation for the increment of fluid content (or $p(\zeta,\, e_{\mathfrak{s}})$)

$$\begin{aligned} \phi &= \phi(e_{\mathfrak{s}},\, \zeta) = \phi_0 + \frac{\phi_0}{K^{\mathfrak{f}}} \left[\alpha\, M - K^{\mathfrak{f}} \right] e_{\mathfrak{s}} + \frac{1}{K^{\mathfrak{f}}} \left[K^{\mathfrak{f}} - \phi_0\, M \right] \zeta \\ &= \phi(e_{\mathfrak{s}},\, e_{\mathfrak{f}}) = \phi_0 + \frac{\phi_0}{K^{\mathfrak{f}}} \left[M(\alpha - \phi_0) \right] e_{\mathfrak{s}} + \frac{\phi_0}{K^{\mathfrak{f}}} \left[M\, \phi_0 - K^{\mathfrak{f}} \right] e_{\mathfrak{f}} \end{aligned}$$

poro-elasticity - the role of porosity

Remark

$$\begin{aligned} \phi &= \phi(e_{\mathfrak{s}},\,\zeta) = \phi_0 + \frac{\phi_0}{K^{\mathfrak{f}}}\left[\alpha\,M - K^{\mathfrak{f}}\right]\,e_{\mathfrak{s}} + \frac{1}{K^{\mathfrak{f}}}\left[K^{\mathfrak{f}} - \phi_0\,M\right]\,\zeta \\ &= \phi(e_{\mathfrak{s}},\,e_{\mathfrak{f}}) = \phi_0 + \frac{\phi_0}{K^{\mathfrak{f}}}\left[M(\alpha - \phi_0)\right]\,e_{\mathfrak{s}} + \frac{\phi_0}{K^{\mathfrak{f}}}\left[M\,\phi_0 - K^{\mathfrak{f}}\right]\,e_{\mathfrak{f}} \end{aligned}$$

porosity (or its change $\phi - \phi_0$) linearly depends on pairs of kinematic variables $e_{\mathfrak{s}}$, $e_{\mathfrak{f}}$, and ζ Porosity $\phi(\mathbf{x},\,t)$ is a dependent field variable. It could be used instead of e.g. ζ or $e_{\mathfrak{s}}$.

poro-elasticity - the role of porosity

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$$\begin{bmatrix} e_{\mathfrak{s}} \\ \phi - \phi_0 \end{bmatrix} = \frac{1}{K}\begin{bmatrix} 1 & \alpha \\ \alpha - \phi_0 & \alpha - \phi_0 \end{bmatrix}\begin{bmatrix} \sigma^M \\ p \end{bmatrix}$$

or it’s inverse

poro-elasticity - the role of porosity

Remark

$$\begin{aligned}\phi &= \phi(e_{\mathfrak{s}},\,\zeta) = \phi_0 + \frac{\phi_0}{K^{\mathfrak{f}}}\left[\alpha\,M - K^{\mathfrak{f}}\right]\,e_{\mathfrak{s}} + \frac{1}{K^{\mathfrak{f}}}\left[K^{\mathfrak{f}} - \phi_0\,M\right]\,\zeta \\ &= \phi(e_{\mathfrak{s}},\,e_{\mathfrak{f}}) = \phi_0 + \frac{\phi_0}{K^{\mathfrak{f}}}\left[M(\alpha-\phi_0)\right]\,e_{\mathfrak{s}} + \frac{\phi_0}{K^{\mathfrak{f}}}\left[M\,\phi_0 - K^{\mathfrak{f}}\right]\,e_{\mathfrak{f}}\end{aligned}$$

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or it’s inverse

$$\begin{bmatrix} \sigma^M \\ p \end{bmatrix} = K^{\mathfrak{s}}\begin{bmatrix} 1 & -\dfrac{\alpha}{\alpha-\phi_0} \\ -1 & \dfrac{1}{\alpha-\phi_0} \end{bmatrix}\begin{bmatrix} e_{\mathfrak{s}} \\ \phi - \phi_0 \end{bmatrix}$$

where only two parameters appear (relation independent of $e_{\mathfrak{f}}$ and ζ).

poro-elasticity - effective stress principle

regarding again our constitutive result (1st line)

$$\begin{bmatrix} \sigma^M \\ \zeta \end{bmatrix} = \begin{bmatrix} K_{\mathrm{u}} - \alpha^2 M & -\alpha \\ \alpha & 1/M \end{bmatrix} \begin{bmatrix} e_{\mathfrak{s}} \\ p \end{bmatrix}$$

or

$$\sigma^M + \alpha\, p = K\, e_{\mathfrak{s}}$$

poro-elasticity - effective stress principle

regarding again our constitutive result (1st line)

$$\begin{bmatrix} \sigma^M \\ \zeta \end{bmatrix} = \begin{bmatrix} K_u - \alpha^2 M & -\alpha \\ \alpha & 1/M \end{bmatrix} \begin{bmatrix} e_{\mathfrak{s}} \\ p \end{bmatrix}$$

or

$$\sigma^M + \alpha\, p = K\, e_{\mathfrak{s}} =: \sigma_E^{M,\mathfrak{s}}$$

i.e., only the weighted balance of mean stress and fluid pressure effectively loads the solid skeleton and causes its volumetric deformation with $0 \le \alpha \le 1$. For $\alpha = 1 - K/K^{\mathfrak{s}} \equiv 1$, it includes Terzaghi's effective stress principle.

poro-elasticity - effective stress principle

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effective stress caused by porosity change

$$\sigma^M + p = \sigma^M + 1\, p = K^{\mathfrak{s}} \frac{1-\alpha}{\alpha - \phi_0} (\phi - \phi_0)$$

i.e. effective stress coefficient is “1”.

poro-elasticity - constitutive equations

the non-equilibrium case:

remember balance of momentum (fluid - quasi-static case)

$$\mathrm{div}(\phi\, p\, \mathbf{I}) = \phi\ \mathrm{grad}\, p + p\ \mathrm{grad}\, \phi = \rho^{\mathfrak{f}}\, \mathbf{b} + \hat{\mathbf{p}}^{\mathfrak{f}}$$

poro-elasticity - constitutive equations

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momentum exchange is undetermined yet!

poro-elasticity - constitutive equations

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momentum exchange is undetermined yet! in general

$$\hat{\mathbf{p}}^{\mathfrak{f}} = \hat{\mathbf{p}}^{\mathfrak{f}}_{eq} + \hat{\mathbf{p}}^{\mathfrak{f}}_{neq}$$

poro-elasticity - constitutive equations

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$$\hat{\mathbf{p}}^{\mathfrak{f}} = \hat{\mathbf{p}}^{\mathfrak{f}}_{eq} + \hat{\mathbf{p}}^{\mathfrak{f}}_{neq}$$

and from CD-inequality we get (for the linear case. . .)

$$\begin{aligned} \hat{\mathbf{p}}^{\mathfrak{f}}_{\mathrm{eq}} &= p\ \mathrm{grad}\, \phi, \\ \hat{\mathbf{p}}^{\mathfrak{f}}_{\mathrm{neq}} &= -\frac{\phi_0^2\, \gamma_0^{\mathfrak{f}R}}{k^{\mathfrak{f}}}\mathbf{w}_{\mathfrak{f}} = -\frac{\phi_0^2\, \eta^{\mathfrak{f}R}}{k^{\mathfrak{s}}}\mathbf{w}_{\mathfrak{f}} \end{aligned}$$

poro-elasticity - constitutive equations

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remember balance of momentum (fluid - quasi-static case)

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$$\begin{aligned} \hat{\mathbf{p}}^{\mathfrak{f}}_{\mathrm{eq}} &= p \operatorname{grad} \phi, \\ \hat{\mathbf{p}}^{\mathfrak{f}}_{\mathrm{neq}} &= -\frac{\phi_0^2\, \gamma_0^{\mathfrak{f}R}}{k^{\mathfrak{f}}} \mathbf{w}_{\mathfrak{f}} = -\frac{\phi_0^2\, \eta^{\mathfrak{f}R}}{k^{\mathfrak{s}}} \mathbf{w}_{\mathfrak{f}} \end{aligned}$$

which gives finally in the linear(ized) case ($\phi \equiv \phi_0$) Darcy's law

$$\operatorname{grad} p = \rho^{\mathfrak{f}R}\, \mathbf{b} - \frac{\phi_0\, \eta^{\mathfrak{f}R}}{k^{\mathfrak{s}}} \mathbf{w}_{\mathfrak{f}}$$

i.e. $\operatorname{grad} p \propto \mathbf{w}_{\mathfrak{f}}$

poro-elasticity - the IBVP

the initial boundary value problem of linear poro-elasticity

Biot's model

equations in the domain, i.e., $\forall\, \mathbf{x} \in \mathcal{B}$

$$
\begin{aligned}
-\text{div}\,(\boldsymbol{\sigma}_E^{\mathfrak{s}} - \alpha\, p\, \mathbf{I}) &= \rho\, \mathbf{b} \\
\frac{\dot{p}}{M} - \frac{k^{\mathfrak{f}}}{\gamma^{\mathfrak{f}R}} \text{div}\,\text{grad}\, p + \alpha\, \text{div}\, \mathbf{v}_{\mathfrak{s}} &= 0
\end{aligned}
$$

Terzaghi's model

$$
\begin{aligned}
-\text{div}\,(\boldsymbol{\sigma}_E^{\mathfrak{s}} - p\, \mathbf{I}) &= \rho\, \mathbf{b} \\
- \frac{k^{\mathfrak{f}}}{\gamma^{\mathfrak{f}R}} \text{div}\,\text{grad}\, p + \text{div}\, \mathbf{v}_{\mathfrak{s}} &= 0
\end{aligned}
$$

boundary conditions, i.e. $\forall\, \mathbf{x} \in \partial\mathcal{B}$

$$
\begin{aligned}
\mathbf{u}_{\mathfrak{s}} &= \bar{\mathbf{u}}_{\mathfrak{s}} \quad \text{on} \quad \Gamma_D^{\mathfrak{s}} \\
p &= \bar{p} \quad \text{on} \quad \Gamma_D^{\mathfrak{f}} \\
\boldsymbol{\sigma} \cdot \mathbf{n} &= \bar{\mathbf{t}} \quad \text{on} \quad \Gamma_N^{\mathfrak{s}} \\
\mathbf{w}_{\mathfrak{f}} \cdot \mathbf{n} &= \bar{w}_{\mathfrak{f}} \quad \text{on} \quad \Gamma_N^{\mathfrak{f}}
\end{aligned}
$$

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\mathbf{w}_{\mathfrak{f}} \cdot \mathbf{n} &= \bar{w}_{\mathfrak{f}} \quad \text{on} \quad \Gamma^{\mathfrak{f}}_N
\end{aligned}
$$

formulation in displacements $\mathbf{u}_{\mathfrak{s}}$ and pressure p - mixed FEM

linear poro-elasticity

the dynamic case - waves

poro-elasticity - dynamic case

(linear) acoustic waves

inertia terms in the balance of momentum have to be included in the consideration! We consider the partial balances of momentum for the fluid and the solid phase (already linearized + const. eqs.)

$$\begin{aligned}
\rho_{11}\,\ddot{\mathbf{u}}_{\mathfrak{s}} + \rho_{12}\,\ddot{\mathbf{u}}_{\mathfrak{f}} + b_0\,F(\dot{\mathbf{u}}_{\mathfrak{s}} - \dot{\mathbf{u}}_{\mathfrak{f}}) &= N\ \mathrm{div\ grad}\,\mathbf{u}_{\mathfrak{s}} + (A+N)\,\mathrm{grad\ div}\,\mathbf{u}_{\mathfrak{s}} \\
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poro-elasticity - dynamic case

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\end{aligned}$$

here, we use a “slightly” modified notation (e.g. for “added mass”)

$$\begin{aligned}
\rho_{11} &= (1-\phi_0)\,\rho^{\mathfrak{s}R} - \rho_{12} \\
\rho_{12} &= (1-\alpha_\infty)\,\phi_0\,\rho^{\mathfrak{f}R} \\
\rho_{22} &= \alpha_\infty\,\phi_0\,\rho^{\mathfrak{f}R}
\end{aligned}$$

no added mass effects for $\alpha \equiv 1$!

poro-elasticity - dynamic case

the (elastic) coefficients

$$
\begin{aligned}
N &= G \\
A &= K - 2\,N/3 + K^{\mathfrak{f}}(1 - \phi_0 - K/K^{\mathfrak{s}})^2/\phi_0^R \\
Q &= \phi_0\,K^{\mathfrak{f}}(1 - \phi_0 - K/K^{\mathfrak{s}})/\phi_0^R \\
R &= \phi_0^2\,K^{\mathfrak{f}}/\phi_0^R \\
P &= A + 2\,N
\end{aligned}
$$

poro-elasticity - dynamic case

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with the “effective porosity”

$$\phi_0^R = \phi_0 + K^{\mathfrak{f}}/K^{\mathfrak{s}}(1-\phi_0 - K/K^{\mathfrak{s}})$$

the tortuosity describing “added mass” (Berryman, 1980)

poro-elasticity - dynamic case

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the tortuosity describing “added mass” (Berryman, 1980)

$\alpha_\infty = 1 - r(1-1/\phi_0),$ with e.g. $r = 1/2$ (for spheres)

and the viscous damping factor

$$b_0 = \eta^{\mathfrak{f}R}\,\phi_0^2/k^{\mathfrak{s}}$$

poro-elasticity - dynamic case

a frequency-dependent correction term (Johnson et al., 1987)

$$F = \sqrt{1 + \frac{1}{2}\, i\, M\, \omega / \omega_{crit}}$$

which takes into account the frequency-dependent momentum interaction (“from viscous to interia”)

poro-elasticity - dynamic case

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$$\omega_{crit} = \frac{\eta^{\mathfrak{f}R}\, \phi_0}{\alpha_\infty\, \rho^{\mathfrak{f}R}\, k^{\mathfrak{s}}} = \frac{\eta^{\mathfrak{f}R}}{\rho^{\mathfrak{f}R}\, R^2} \quad \text{and} \quad \frac{\omega}{\omega_{crit}} = \text{Wo}^2$$

the red expression is for cylindrical tubes (with diameter $2\,R$)

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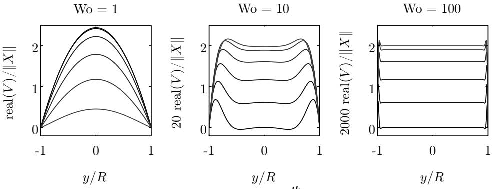
typical velocity profiles in tubes



poro-elasticity - dispersion relation

physical behaviour of waves in poro-elastic media

$$
\begin{aligned}
\rho_{11}\,\ddot{\mathbf{u}}_{\mathfrak{s}} + \rho_{12}\,\ddot{\mathbf{u}}_{\mathfrak{f}} + b_0\,F(\dot{\mathbf{u}}_{\mathfrak{s}} - \dot{\mathbf{u}}_{\mathfrak{f}}) &= N\ \mathrm{div}\ \mathrm{grad}\,\mathbf{u}_{\mathfrak{s}} + (A+N)\mathrm{grad}\ \mathrm{div}\,\mathbf{u}_{\mathfrak{s}} \\
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$$

standard harmonic ansatz for harmonic waves

$$
\begin{aligned}
\mathbf{u}_{\mathfrak{s}}(\mathbf{x},t) &= \hat{\mathbf{u}}_{\mathfrak{s}}(\mathbf{x},\omega)\ \exp(i\,\omega\,t), \\
\mathbf{u}_{\mathfrak{f}}(\mathbf{x},t) &= \hat{\mathbf{u}}_{\mathfrak{f}}(\mathbf{x},\omega)\ \exp(i\,\omega\,t),
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poro-elasticity - dispersion relation

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and splitting in transversal and longitudinal part

$$\begin{aligned}\hat{\mathbf{u}}_{\mathfrak{s}}(\mathbf{x},\omega) &= \mathrm{grad}\,\phi_{\mathfrak{s}}(\mathbf{x},\omega) + \mathrm{rot}\,\boldsymbol{\psi}^{\mathfrak{s}}(\mathbf{x},\omega), \\ \hat{\mathbf{u}}_{\mathfrak{f}}(\mathbf{x},\omega) &= \mathrm{grad}\,\phi_{\mathfrak{f}}(\mathbf{x},\omega) + \mathrm{rot}\,\boldsymbol{\psi}^{\mathfrak{f}}(\mathbf{x},\omega),\end{aligned}$$

poro-elasticity - dispersion relation

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with vector-values potentials

$$\begin{aligned} \phi_\alpha(\mathbf{x},\omega) &= \tilde{\phi}_\alpha(k,\omega)\ \exp(i\,k\,\mathbf{x}), \\ \boldsymbol{\psi}_\alpha(\mathbf{x},\omega) &= \tilde{\boldsymbol{\psi}}_\alpha(k,\omega)\ \exp(i\,k\,\mathbf{x}) \end{aligned}$$

poro-elasticity - transversal mode

shear waves

dispersion relation for shear waves (after some algebra)

$$\mathbf{A}_S \, \tilde{\boldsymbol{\Psi}} = k^2 \, \mathbf{B}_S \, \tilde{\boldsymbol{\Psi}},$$

(generalized eigenvalue problem - could be analytically/numerically solved) with $\tilde{\boldsymbol{\Psi}} = [\tilde{\psi}_{\mathfrak{s}} \; \tilde{\psi}_{\mathfrak{f}}]^T$ and

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$$\mathbf{A}_S = \begin{bmatrix} \tilde{\rho}_{11} & \tilde{\rho}_{12} \\ \tilde{\rho}_{21} & \tilde{\rho}_{22} \end{bmatrix} \omega^2 \qquad \text{and} \qquad \mathbf{B}_S = \begin{bmatrix} N & 0 \\ 0 & 0 \end{bmatrix}$$

poro-elasticity - transversal mode

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with one complex solution for the the wave number $k(\omega)$ and $\xi = k^2$

$$\xi = \frac{\tilde{\rho}_{11}\,\tilde{\rho}_{22} - \tilde{\rho}_{12}^2}{N\,\tilde{\rho}_{22}}.$$

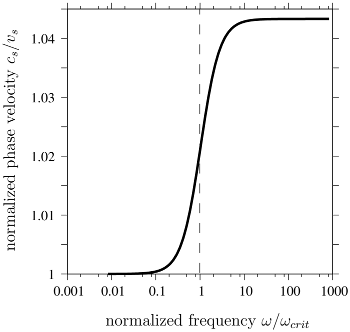
and the complex densities (as abbreviations. . .)

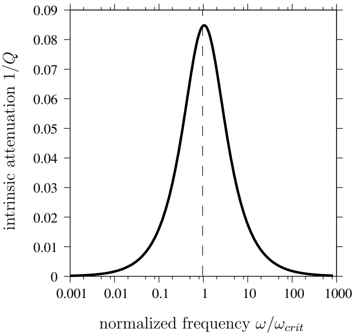
$$\tilde{\rho}_{12} = \rho_{12} + i\,b_0\,F/\omega, \quad \tilde{\rho}_{11} = \rho_{11} - i\,b_0\,F/\omega, \quad \tilde{\rho}_{22} = \rho_{22} - i\,b_0\,F/\omega.$$

poro-elasticity - transversal mode

shear waves (here for Berea sandstone)

phase velocity $c = 1/\mathrm{Re}(k)$
intrinsic attenuation $1/Q = 2\,|\mathrm{Im}(k)/\mathrm{Re}(k)|$





poro-elasticity - longitudinal mode

compressional waves

dispersion relation for compressional waves (after some algebra)

$$\mathbf{A}_P\,\tilde{\mathbf{\Phi}} = k^2\,\mathbf{B}_P\,\tilde{\mathbf{\Phi}},$$

(generalized eigenvalue problem - could be analytically/numerically solved) with $\tilde{\mathbf{\Phi}} = [\tilde{\phi}_{\mathfrak{s}}\ \tilde{\phi}_{\mathfrak{f}}]^T$ and

poro-elasticity - longitudinal mode

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poro-elasticity - longitudinal mode

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quadratic equation for $\xi = k^2$ with two physical solutions for the complex wave number $k(\omega)$

$$\xi_{1,2} = \frac{\Delta \pm \sqrt{\Delta^2 - 4\,(P\,R - Q^2)\,(\tilde{\rho}_{11}\,\tilde{\rho}_{22} - \tilde{\rho}_{12}\,\tilde{\rho}_{12})}}{2\,(P\,R - Q^2)}$$

with

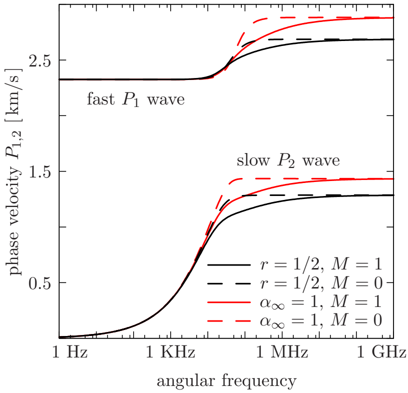
$$\Delta = P\,\tilde{\rho}_{22} + R\,\tilde{\rho}_{11} - 2\,Q\,\tilde{\rho}_{12}$$

i.e. the fast P-wave (P) and the slow P-wave (Biot-wave)

poro-elasticity - longitudinal mode

compressional waves (here for a high porous bone)

phase velocity of $P_{1,2}$ mode with $c_{1,2} = 1/\text{Re}(k_{1,2})$

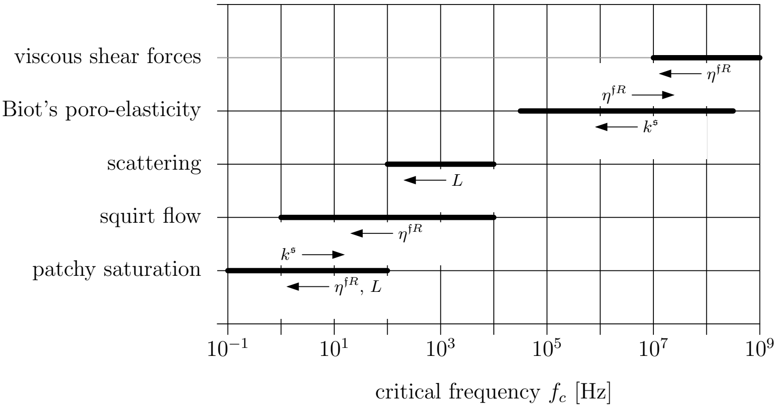


linear poro-elasticity

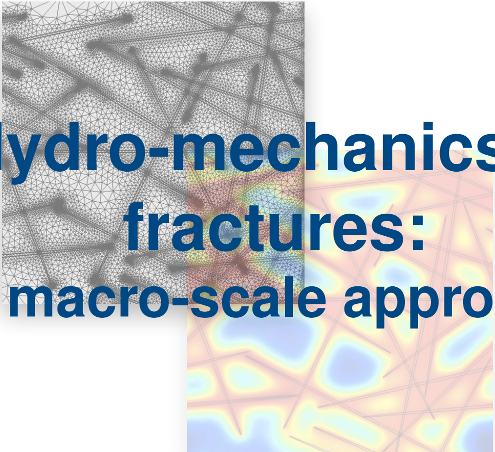
the role of heterogeneities

the role of heterogeneities

motivation: from homogeneous to heterogeneous porous media



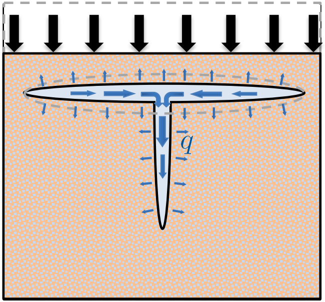
fluid properties ($\eta^{\mathfrak{f}R}$) and characteristic lengths (L, $k^{\mathfrak{s}}$) matter



Hydro-mechanics of fractures: a macro-scale approach

example: uniaxial compression

uniaxial compression:



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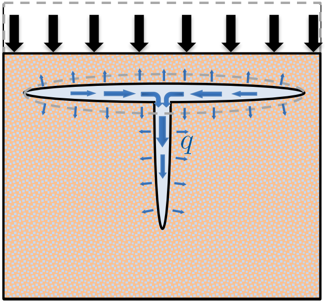

* pressure gradients induce **viscous fluid flow - attenuation**

example: uniaxial compression

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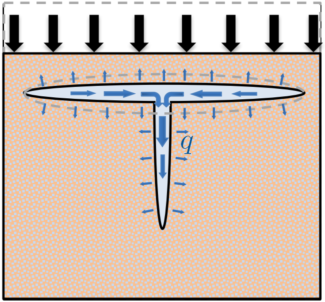

* pressure gradients induce **viscous fluid flow - attenuation**
* mesoscopic scale (along vertical fracture)

$\Rightarrow$ **fracture flow**

example: uniaxial compression

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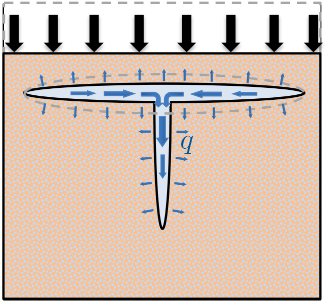


* pressure gradients induce **viscous fluid flow - attenuation**
* mesoscopic scale (along vertical fracture)

  ⇒ **fracture flow**

* microscopic scale (from inclusions towards the porous matrix)

  ⇒ **leak-off**

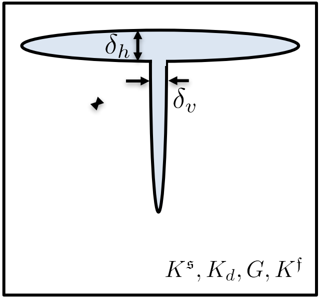
example: uniaxial compression

flow processes are controlled:

* **transmissivity** of fracture

  (aperture δ, viscosity $\eta^{\mathfrak{f}R}$, fracture stiffness)

* **permeability** of rock matrix $k^{\mathfrak{s}}$



Energy dissipation processes at different characteristic times (frequencies) and with different magnitude

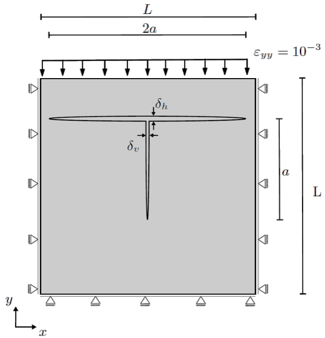
ATTENUATION CAUSED BY FRACTURES

effects of **high aspect ratio inclusions** on effective attenuation

example: two connected fractures - IBVP

assumptions of the coupled FEM analysis:

- 2-dimensional domain (plane strain)
- fully saturated porous matrix
- two interconnected fractures
- uniaxial compression: $\varepsilon_{yy} = 10^{-3}$ (small strains)



geometrical parameters:

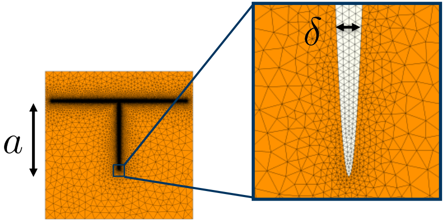
- fracture half-length: $a = 5$ cm
- domain length: $L = 2.2\,a$

- aperture horizontal crack: $\delta_h(t_0) = 100\ \mu$m
- aperture vertical crack: $\delta_v(t_0) = 5\ldots500\ \mu$m

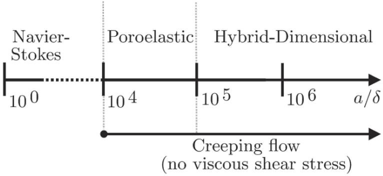
example: two connected fractures - IBVP

limitations during numerical modeling (FEM):

- spatial discretization of **thin crack geometry**
- **meshing**
- large number of elements (DOFs)

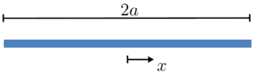


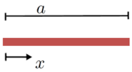
- hybrid-dimensional approach (Vinci et al., *GRL*, 2014; *WRR*, 2014; *GJI*, 2015) & Schmidt and Steeb, *GEM*, 2019
- fluid flow through high-aspect-ratio geometries

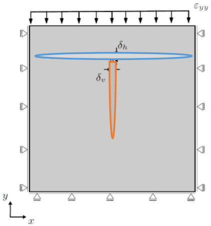


hybrid dimensional approach

* flow through porous matrix $\Longrightarrow$ 2-D poroelastic problem
* flow along fractures $\Longrightarrow$ 1-D, compressible, viscous fluid







conservation of mass & momentum $\Longrightarrow$ Poiseuille flow

$$\frac{\partial p}{\partial t} - \underbrace{\frac{\partial}{\partial x}\left(\frac{\delta^2}{\beta_{\mathfrak{f}}\, 12\eta}\frac{\partial p}{\partial x}\right)}_{\text{diffusion}} - \underbrace{\left(\frac{\delta}{\beta_{\mathfrak{f}}\, 12\eta}\frac{\partial \delta}{\partial x}\right)\frac{\partial p}{\partial x}}_{\text{nozzling}} - \underbrace{\frac{\delta^2}{12\eta}\left(\frac{\partial p}{\partial x}\right)^2}_{\text{quadratic}} = - \underbrace{\frac{1}{\beta_{\mathfrak{f}}\,\delta}\frac{\partial \delta}{\partial t}}_{\text{RHS (hydro-mechanical coupling)}}$$

diffusion **nozzling** **quadratic** **RHS** (hydro-mechanical coupling)

hybrid dimensional approach

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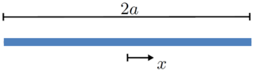

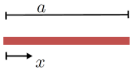

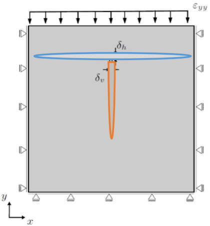

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numerical quality **independent** of fractures' aspect ratio

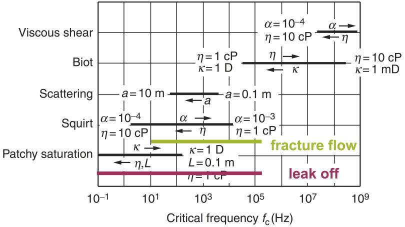
hybrid dimensional approach

macroscopic variables obtained through volume averaging

$$\boldsymbol{\sigma}_M = \frac{1}{V_m}\int_{V_m} \boldsymbol{\sigma}_m \mathrm{d}v \qquad \boldsymbol{\varepsilon}_M = \frac{1}{V_m}\int_{V_m} \boldsymbol{\varepsilon}_m \mathrm{d}v$$

FFT & complex elastic modulus, e.g.

$$M(\omega) = \frac{\sigma_{yy}(\omega)}{\varepsilon_{yy}(\omega)}$$



quantification of dissipation through the inverse quality factor

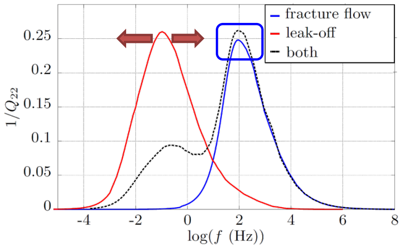
$$\frac{1}{Q_{yy}} = \frac{\mathrm{Im}(M(\omega))}{\mathrm{Re}(M(\omega))}$$

* amplitude
* characteristic frequencies

hybrid dimensional approach

characteristic frequencies

- fracture flow $f_c \approx 10^2$ Hz
- leak-off $f_c \approx 10^{-1}$ Hz



amplitudes

- leak-off affected by faster fracture flow process

$$k^{\mathfrak{s}} = 10^{-18} \ldots 10^{-30}\,\text{m}^2$$

$$\delta_v(t_0) = 10\,\mu\text{m}$$

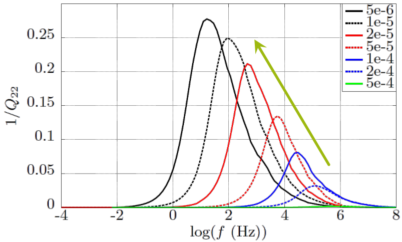
leak-off $\Longrightarrow$ rock permeability

fracture flow $\Longrightarrow$ fracture transmissivity $\Longrightarrow$ fracture aspect ratio

hybrid dimensional approach

aspect ratio of fracture (only squirt-flow):

* $k^{\mathfrak{s}} = 10^{-30}\,\mathrm{m}^2$
  (impermeable matrix)
* $a = 5$ cm $\quad \delta_h(t_0) = 100\,\mu\mathrm{m}$
* $\delta_v(t_0) = 5\,\mu\mathrm{m} \ldots 500\,\mu\mathrm{m}$
* aspect ratio: $a/\delta = 10^2 \ldots 10^4$



increasing aspect ratio:

* higher amplitude
  (more dissipation)
* lower frequency
  (slower process)

mesoscopic loss is of relevance for REVs with high-aspect-ratio inclusions

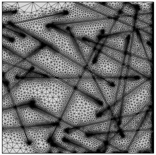
take home message

linear poro-elasticity / theory

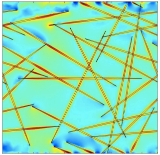
* Biot's theory could be derived from mixture theory
* "formal" linearization
* the role of porosity / state variables

attenuation

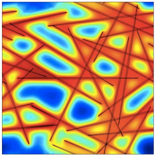
* Biot's wave (2nd P wave) is highly attenuated
* heterogeneities = attenuation
* fractures



FE discretization



pressure distribution ($t \rightarrow t_0$)



pressure distribution ($t > t_0$)

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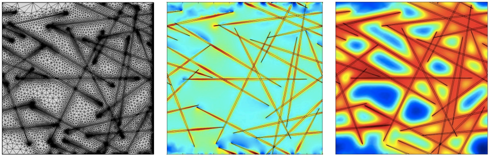

FE discretization | pressure distribution ($t \rightarrow t_0$) | pressure distribution ($t > t_0$)

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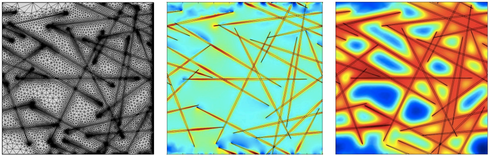

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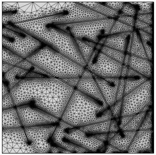

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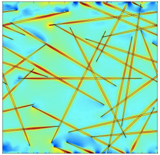

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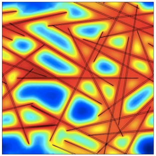

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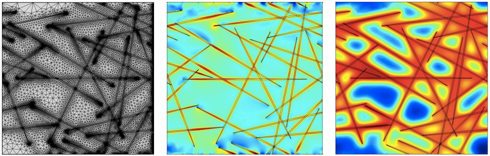

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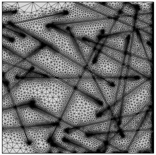

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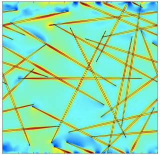

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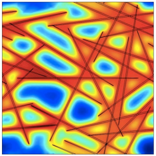

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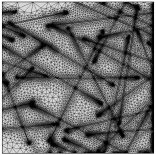
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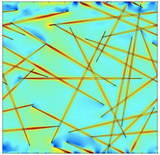
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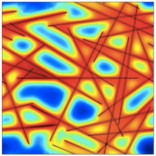
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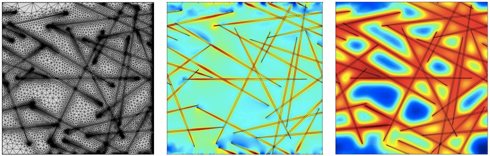

FE discretization | pressure distribution ($t \to t_0$) | pressure distribution ($t > t_0$)

. . . thanks for listening. . .

and have a safe trip home!