BREAKAGE MECHANICS

statistical constitutive approach from micro to macro

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PROPOSED PHILOSOPHY

› Energy considerations at micro-scale
  - energy **scaling** (e.g. with grain sizes)
  - rate of dissipation via **redistribution of strain energy**

› Statistical homogenisation
  - use of **grainsize distribution** for averaging strain energy
  - simplified grainsize distribution via scalar **Breakage**

› Use thermodynamics to derive **macro-scale** constitutive laws
BREAKAGE DEFINITION

\[ B = \frac{B_t}{B_p} \]

Percent finer, %

Grain size, \( d \) (log scale)

\( B_t \)

\( B_p \)

Current distribution, \( F \)

Ultimate distribution, \( F_u \)

Initial distribution, \( F_0 \)
equivalence hypothesis

\[ p(B, d) = p_0(d)(1 - B) + p_u(d)B \]

Coop et al., 2004
energy scaling

\[ \Psi \equiv \int p(B,x)\psi(x,\text{state})\,dx = \left( \int p(B,x)f_\psi(x)\,dx \right) \cdot \psi_r(\text{state}) \]

\[ \Psi \equiv (1 - \vartheta B)\psi_r(\text{state}) \]

Force chains in compacted granular assemblies using DEM: (left) bi-modal distribution \((d_{\text{large}}=10.d_{\text{small}})\), and (right) uniform distribution (by size).
energy scaling

\[ \Psi \equiv \int p(B, x) \psi(x, \text{state}) \, dx = \left( \int p(B, x) f_\psi(x) \, dx \right) \cdot \psi_r(\text{state}) \]

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Force chains in compacted granular assemblies using DEM: (left) bi-modal distribution \((d_{\text{large}}=10, d_{\text{small}})\), and (right) uniform distribution (by size).
THERMODYNAMICS STRESSES

and residual breakage energy

\[ E^*_B = E_B (1 - B) = \Psi - \Psi_u \]

stress – elastic strain

\[ \sigma_{ij} = \frac{\partial \Psi}{\partial \varepsilon_{ij}^e} \]

breakage energy – breakage

\[ E_B = - \frac{\partial \Psi}{\partial B} = \Psi_0 - \Psi_u \]
BREAKEAGE CRITERION

\[
\begin{align*}
\gamma_B(B, E_B) &= E_B(1-B)^2 - E_c \leq 0 \\
\delta B &= \delta \lambda \frac{\partial \gamma_B}{\partial E_B} \\
E_B &= \frac{\partial \Psi}{\partial B} \quad \text{and} \quad p = \frac{\partial \Psi}{\partial \varepsilon^e} \quad \text{so e.g. when} \quad \Psi = \frac{1}{2} (1 - \vartheta B) K \varepsilon^e^2 \\
E_B^* &= E_B (1 - B) \\
E_c &= \text{critical breakage energy}
\end{align*}
\]
connection to established theory

fracture mechanics

\[ \sigma_{cr} = \sqrt{\frac{EG_c}{\pi a}} \]

breakage mechanics

\[ P_{cr} = \sqrt{\frac{2KE_c}{\vartheta}} \]

\[ P_{cr} = P_r \left(\frac{3KE_c}{4\vartheta P_r}\right)^{2/3} \]

E<sub>c</sub> = critical breakage energy constant
\( \vartheta \) = normalised surface area
**MODES OF DISSIPATION**

**compression**

**Active:** dissipation by freeing surface energy

**Passive:** redistribution of locked-in strain energy

\[
(a) \quad D_B = E_B \delta B
\]

**Passive:** Configurational dissipation (volumetric plastic dissipation) \( D_v \)

**distortion**

**Active:** plastic dissipation (Coulomb/distortional \( D_d \equiv Mp |\dot{\varepsilon}_s^p| \geq 0 \))

**Passive:** breakage/abrasion \( D_B = E_B \delta B \)
dissipation, yield, flow, and critical state (1st go)

\[ y_B = \frac{E_B}{E_c} (1 - B)^2 + \left( \frac{q}{M_p} \right)^2 - 1 \leq 0 \]

\[ \dot{B} = \lambda (1 - B)^2 \]

\[ \Psi = \frac{1}{2} (1 - \partial B) \left( K \varepsilon_v^2 + 3 G \varepsilon_s^2 \right) \]

\[ \Phi = \sqrt{D_B^2 + D_d^2 + D_v^2} \]
MULTI-COMPONENT THEORY

Effect on crushing strength

\[ \sum_{i=1}^{N} f^{(i)} = 1 \]

\[ p^{(i)}(D) = p_0^{(i)}(D)(1 - B^{(i)}) + p_u^{(i)}(D)B^{(i)} \]

\[ \Psi = \sum_{i=1}^{N} \psi_r^{(i)}(\varepsilon)f^{(i)}(1 - \Theta^{(i)}B^{(i)}) \]

\[ \Phi_B = \sum_{i=1}^{N} f^{(i)}E_B^{(i)} \delta B^{(i)} \geq 0 \]

Yielding pressure against the (strong) quartz mass fraction

Breakage evolution in the weak (calcareous) and strong (quartz) phases, against pressure

\begin{align*}
\text{Experiments- calcareaous} & \quad \text{Experiments- quartz} \\
\text{Theory- calcareaous} & \quad \text{Theory- quartz} \\
\end{align*}
Bi-mixture of grains (breakage) and cement (damage)
Bi-mixture of grains (breakage) and cement (damage)

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Adamswiller</th>
<th>Bentheim</th>
<th>Calcarenite</th>
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<tbody>
<tr>
<td>$G_g$ and $G_c$</td>
<td>3200 MPa</td>
<td>7588 MPa</td>
<td>75,500 kPa</td>
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<td>$K_g$ and $K_c$</td>
<td>20,000</td>
<td>42,000</td>
<td>3470</td>
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<tr>
<td>$M$</td>
<td>1.5</td>
<td>1.7</td>
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<tr>
<td>$E_{BC}$</td>
<td>2.6 MPa</td>
<td>3.85 MPa</td>
<td>15 kPa</td>
</tr>
<tr>
<td>$E_{DC}$</td>
<td>1.6 MPa</td>
<td>3.0 MPa</td>
<td>18 kPa</td>
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<tr>
<td>$c$</td>
<td>40 MPa</td>
<td>60 MPa</td>
<td>300 kPa</td>
</tr>
<tr>
<td>$\phi_g$</td>
<td>40°</td>
<td>70°</td>
<td>78°</td>
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</table>
Bi-mixture of grains (mechanics) and partial saturation (hydro)

\[ W \equiv \sigma_{ij} \dot{\varepsilon}_{ij} - \phi s \dot{S}_r \]

\[ \Psi \equiv (1 - \vartheta_M B) \psi_r (\varepsilon_{ij}^e) + (1 + \vartheta_H B) \psi_r (S_r) \]
‘NON-UNIQUE CRITICAL STATE’

The problem (phenomenologically)

The reason (micromechanically)
POROSITY AND STRAINS

\[ \dot{\phi}^e = -c(1 - \phi)\dot{\varepsilon}_v^e \]
\[ \dot{\phi}^p = -(1 - \phi)\dot{\varepsilon}_v^p \]
\[ c = (2\nu_g)^{-\frac{1}{\phi}} \]

\[ \nu_g = 0.5 \quad \nu_g = 0 \]

\[ \dot{\varepsilon}_v = -\frac{\phi}{(1 - \phi)} \quad (\text{soil mechanics}) \]
\[ \dot{\varepsilon}_v^p = -\frac{\phi}{(1 - \phi)} \quad (\text{and } \dot{\phi}^e = 0) \]

\[ \nu_g = 0 \quad \nu_g = 0.5 \]

\[ V_S^{II} = V_S^I \]

Before (I)

After (II)

Plastic

Elastic
RELATIVE POROSITY

\[ \tau = \frac{\phi_{\text{max}}(B) - \phi}{\phi_{\text{max}}(B) - \phi_{\text{min}}(B)} \]

\[ \phi_{\text{max}} = \alpha_u (1 - B)^u \]
\[ \phi_{\text{min}} = \alpha_l (1 - B)^l \]

Crushed basalt (Youd, 1973)
Sand mixture (Youd, 1973)
NEGATIVE WORK AND DISSIPATION

\[ \Phi \equiv \sqrt{D_B^2 + D_\phi^2 + D_s^2} + r_B D_B + r_\phi D_\phi \]

\[
D_B = \frac{\sqrt{E_B E_C}}{(1 - B) \cos(\omega)} \frac{\dot{B}^2}{\langle \dot{B} \rangle}
\]

\[
D_\phi = \frac{\sqrt{E_B E_C}}{(1 - B) \sin(\omega)} \frac{E_\phi}{E_B} \dot{\phi}
\]

\[ D_s = M p \dot{\varepsilon}_s \]

during breakage \( \dot{B} > 0 \):

\[ \frac{E_\phi}{E_B} \frac{\dot{\phi}}{\dot{B}} = \tan(\omega) \]

Dissip. via pore collapse

Dissip. via breakage

\[ = \tan(\omega) \]

\[ r_B = \gamma \tau \cos(\omega) \]

\[ r_\phi = \gamma \tau \sin(\omega) \]
dissipation, yield, flow, and critical state (2\textsuperscript{nd} go)

\[
y_B = \left( \frac{E_B}{E_c} (1 - B) - \gamma \tau \right)^2 + \left( \frac{q}{M_p} \right)^2 - 1 \leq 0
\]

\[
\Phi \equiv \sqrt{D_B^2 + D_{\phi}^2 + D_s^2} + r_B D_B + r_{\phi} D_{\phi}
\]

elastic potential

critical state

dissipation

breakage

dilation
dissipation, yield, flow, and critical state (2nd go)

\[
y_B = \sqrt{\frac{E_B}{E_c} (1 - B) - \gamma \tau} + \left( \frac{q}{Mp} \right)^2 - 1 \leq 0
\]

\[
\dot{\varepsilon}_p^s = 2\lambda \frac{q}{(Mp)^2}, \quad \dot{\varepsilon} = 2\lambda F_\text{CS} = \frac{1 - B}{\sqrt{E_B E_c}} \cos^2(\omega), \quad \dot{\phi} = 2\lambda \frac{1 - B}{E_B E_c} E_B \sin^2(\omega)
\]

\[
F \equiv F_\text{CS} = \sqrt{\frac{E_B}{E_c}} (1 - B) - \gamma \tau = 0
\]

\[
\Phi \equiv \sqrt{D_B^2 + D_\phi^2 + D_s^2} + r_B D_B + r_\phi D_\phi
\]

Critical state for humans

Elastic potential

breakage
dilation

"for humans"
MODEL OF BREAKAGE AND POROSITY

‘non-unique’ critical state
‘non-unique’ critical state

Tests by Bandini-Coop (2012):

- $B_r = 0.14 - 0.23$
- $B_r = 0.31 - 0.35$
- $B_r = 0.42 - 0.47$

<table>
<thead>
<tr>
<th>Index properties</th>
<th>$\alpha_u$</th>
<th>$\alpha_l$</th>
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<th>$l$</th>
<th>$\theta$</th>
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<table>
<thead>
<tr>
<th>Mechanical parameters</th>
<th>$\bar{K}$</th>
<th>$\bar{G}$</th>
<th>$M$</th>
<th>$E_C$ [kPa]</th>
<th>$\gamma$</th>
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<td>10000</td>
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<td>5</td>
<td>0.95</td>
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</tbody>
</table>

only ‘free’ parameter
Micro to Macro via energy scaling and redistribution, and statistical homogenisation

Breakage mechanics used to get GSD in space and time

Porosity is internal variable (not total plastic strain!)

Clearer link of porosity rate to strain rate

Introducing relative porosity dependent on GSD

Predicting critical state, preconsolidation pressure, yield surface and wetting collapse rather than imposing them