A phase field modeling approach to unsaturated poromechanics

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Outline

- Motivation

  Effects of air-liquid interfaces on localised deformations of a porous skeleton (and viceversa)

- Formulation of the phase field model

  A Cahn-Hilliard (CH) energy for a biphasic fluid, within the framework of gradient poromechanics

- Thermodynamical restrictions

  - Constitutive relations for the biphasic fluid, (air-water) mixture
  - Constitutive relation for the capillary pressure (wetting of the grains)
  - Constitutive relations for the solid
  - Generalised Darcy law

- Asymptotic analysis of a partially saturated undeformable column

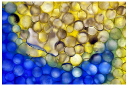
  Equilibria of the biphasic fluid within the region of partial saturation

- Conclusions & further developments

Solid-fluid micro-scale interactions

Multi-phase flow

1. of a non-wetting fluid (drying), capillary trapping



may ⇒ cause **damaging**

Image: M. Szulczewski, MIT

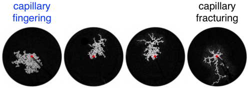
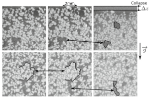


Image: Holtzman et al (2012) PRL

2. of a wetting fluid (imbibition), soil imbibition



may cause irreversible deformations (volumetric collapse)

Image: Bruchon et al (2013) Gr.Mat.

Modeling the effects of capillary fingering on the skeleton microstructure

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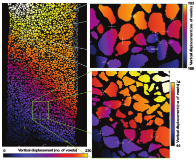
ADVANCED MODELING OF MICRO-SCALE INTERACTIONS

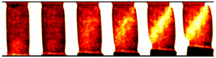
⇒

Microstructure evolution

Localised deformations may cause localised changes of porosity and permeability



Vertical displacement



Porosity

Image: Hall et al. (2010) Géotech.

Modeling of damage occurrence and its interaction with drainage and imbibition processes.

Gradient theory of (thermo-)poromechanics

The forces which can be balanced by gradient media are prescribed by the external working:

$$W^{\text{ext}}\left(v^s, v^f\right) = \sum_c \left\{ \int_{\mathcal{D}} b_\alpha^c \, v_\alpha^c + \int_{\partial\mathcal{D}} \left(t_\alpha^c \, v_\alpha^c + \underbrace{\tau_\alpha^c \, v_{\alpha,\beta}^c \, m_\beta}_{\substack{\text{double forces}\\ \text{(couples)}}} \right) + \int_{\mathcal{E}} \underbrace{f_\alpha^c \, v_\alpha^c}_{\text{edge forces}} \right\}$$

where tractions satisfy the generalised Cauchy theorem, $c = \{s, f\}$

$$\begin{aligned} \left(\Sigma_{\alpha\beta}^c - \Pi_{\alpha\beta\gamma,\gamma}^c\right) m_\beta - \left(\mathcal{Q}_{B\beta}\Pi_{\alpha\beta\gamma}^c m_\gamma\right)_{,B} &= t_\alpha^c, && \text{on } \partial\mathcal{D} \\ \Pi_{\alpha\beta\gamma}^c m_\gamma m_\beta &= \tau_\alpha^c && \text{on } \partial\mathcal{D} \\ [\![\mathcal{Q}_{B\beta}\Pi_{\alpha\beta\gamma}^c m_\gamma \mu_B]\!] &= f_\alpha^c && \text{on } \mathcal{E} \end{aligned}$$

and the overall porous material is balanced.

Gradient theory of (thermo-)poromechanics

1$^{\text{st}}$ and 2$^{\text{nd}}$ principle of thermodynamics

⇓

External working for a gradient solid-fluid mixture

overall ⇒⇒⇒ equilibrium

Skeleton strain working W_{int}

$$W_{\text{ext}} = \sum_{k=1}^{2} \left[\int_{\partial\mathcal{D}^*} t_\alpha^k \dot{\chi}_\alpha^k + \tau_\alpha^k \frac{\partial \dot{\chi}_\alpha^k}{\partial n} + \int_{\mathcal{D}^*} b_\alpha^k \dot{\chi}_\alpha^k + \int_{\partial\partial\mathcal{D}^*} f_\alpha^k \dot{\chi}_\alpha^k \right]$$

⇓

Clausius-Duhem inequality

⇓

Thermodynamical restrictions

Solid dissipation $\Phi_s = 0$	Fluid dissipation $\Phi_f \geq 0$	Thermal dissipation $\Phi_{th} \geq 0$
Consititutive law of the porous skeleton	Generalised Darcy's law	Fourier law

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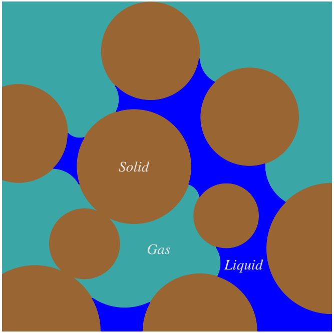
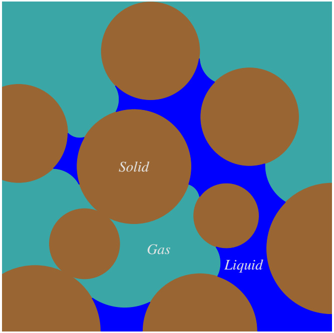
Thermodynamical restrictions: unsaturated free energy



Vanishing of solid dissipation $\Phi_s = 0$ yields the Lagrangian energy of the overall medium

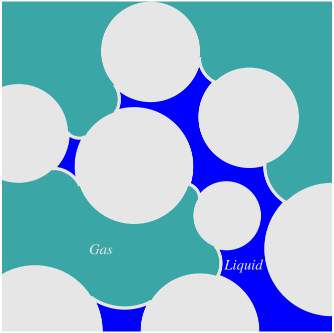
Thermodynamical restrictions: unsaturated free energy



Vanishing of solid dissipation $\Phi_s = 0$ yields the Lagrangian energy of the overall medium

$$\Psi = \Psi_{\mathsf{S\&I}} + \phi\,\rho_w S_w\,\psi_f^I$$

Thermodynamical restrictions: unsaturated free energy



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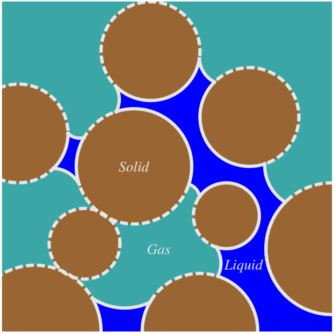
Biphasic fluid



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Solid & Interfaces = solid grains + interfaces

Biphasic fluid

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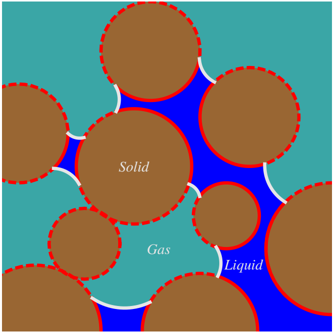

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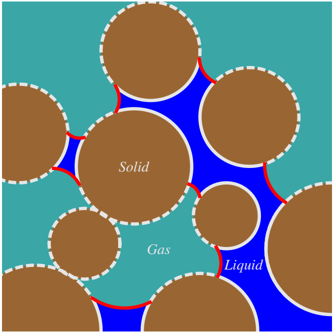

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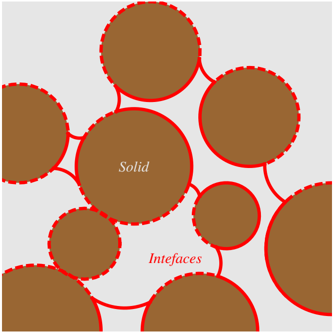

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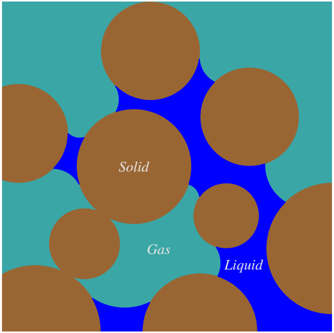

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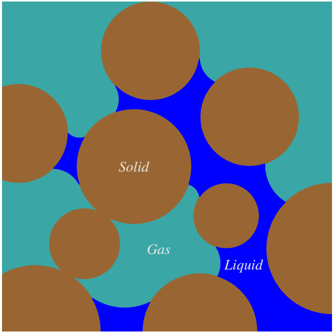
Solid & Interfaces = solid grains + interfaces

Biphasic fluid

Thermodynamics $\Rightarrow$ $\Psi_{\mathsf{s\&I}} = \Psi_{\mathsf{s\&I}}\left(E_{ij}, E_{ij,k}, \phi, \phi S_w, (\phi S_w)_{,i}\right)$

Thermodynamical restrictions: unsaturated free energy



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Solid & Interfaces = solid grains + interfaces

Biphasic fluid

Thermodynamics $\Rightarrow$ $\Psi_{\text{S\&I}} = \Psi_{\text{S\&I}}\left(E_{ij}, E_{ij,k}, \phi, \phi S_w, (\phi S_w)_{,i}\right)$

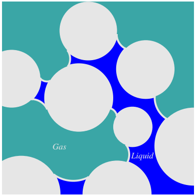
$$\Psi_{\text{S\&I}} = \Psi_B\left(E_{ij}, \phi\right) + \phi U\left(S_w\right) + \Psi_{nl}\left(E_{ij,k}, (\phi S_w)_{,i}\right);$$

Biot-like energy; Capillary energy; Non-local energy

$$\Psi_f = \rho_w S_w \psi_f^I$$

van der Waals-like double-well energy

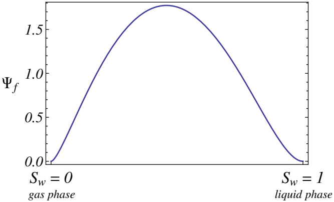
Free energies & constitutive relations - fluid



The biphasic fluid energy, Ψ_f

The fluid is constituted by a liquid & a gas phase

$$\psi_f = \psi_f^I\left(\frac{1}{\rho_f}\right), \quad \mathcal{P} = -\frac{\partial \psi_f^I}{\partial(1/\rho_f)}$$ thermodynamic pressure



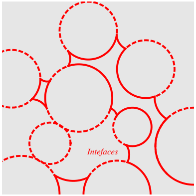
$$\rho_f = \rho_w S_w, \quad \Psi_f = \rho_w S_w \psi_f^I, \quad \mu = \frac{\partial \Psi_f}{\partial S_w}$$

The two phases $S_w = 0/1$ coexist at equilibrium at atmospheric pressure ($\mu = 0$) as they are iso-potential minima of the energy (Maxwell's rule).

Fluid interfaces, Ψ_{nl}

The nonlocal energy accounts for the formation and the displacement of gas-liquid & solid-fluid interfaces in the pores:

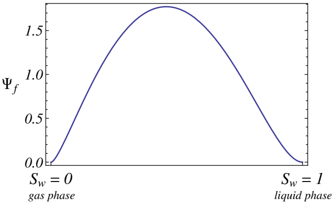
$$\gamma_l = -\phi S_w \frac{\partial \Psi_{nl}}{\partial (\phi S_w)_{,l}}$$ fluid hyper-stress

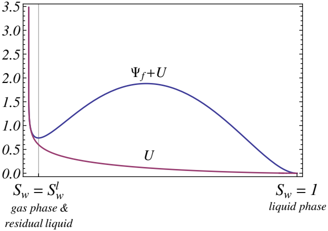


The capillary energy U accounts for the energy stored into the solid-fluid and liquid-gas interfaces

$$\phi\,\mathcal{P}_c = -\frac{\partial \Psi_{\mathsf{S\&l}}}{\partial S_w} = -\phi\,\frac{d\,U}{d\,S_w}$$

In the porous medium $\Psi_f + U$ is the effective energy of the pore fluid





Now the two phases $S_w = S_w^l/1$ do not coexist at equilibrium at atmospheric pressure $(\mu = \frac{\partial \Psi_f}{\partial S_w} = 0)$: for coexistence we need suction $\Psi_f + U - \mu_c\,S_w$, $\mu_c < 0$.

Free energies & constitutive relations - Solid

The porous skeleton, Ψ_B, U

$$S_{ij} = \frac{\partial \Psi_{\mathsf{S\&I}}}{\partial E_{ij}} = \frac{\partial \Psi_B}{\partial E_{ij}}, \quad \mathcal{P} - S_w \mathcal{P}_c = \frac{\partial \Psi_{\mathsf{S\&I}}}{\partial \phi} = \frac{\partial \Psi_B}{\partial \phi} + U$$

Solid interfaces, Ψ_{nl}

The nonlocal energy accounts for the formation and the displacement of solid-solid & solid-fluid interfaces:

$$P_{ijk} = \frac{\partial \Psi_{nl}}{\partial E_{ij,k}}$$ solid hyper-stress

Assuming incompressibility of the solid grains (& small deformations):

Generalised Bishop stresses

$$S'_{ij} = S_{ij} + (\mathcal{P} - \mathcal{P}_c\, S_w)\, \delta_{ij}, \quad P'_{ijk} = P_{ijk} - \delta_{ij}\gamma_k$$

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Thermodynamical restrictions: generalised Darcy's law

$$-\frac{1}{S_w}\mathcal{P}_{,k} + \left[\quad \mathcal{P}_c \quad - \left(\frac{\gamma_l}{\phi S_w}\right)_{,l} \right]_{,k} = A_{kl} w_l - \frac{b_k^{0f}}{\phi S_w}$$

Thermodynamical restrictions: generalised Darcy's law

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thermodynamic pressure

Thermodynamical restrictions: generalised Darcy's law

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thermodynamic pressure | capillary pressure

Thermodynamical restrictions: generalised Darcy's law

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thermodynamic pressure — capillary pressure — fluid hyper-stress

Thermodynamical restrictions: generalised Darcy's law

$$-\frac{1}{S_w}\mathcal{P}_{,k} + \left[\ \mathcal{P}_c \ - \left(\frac{\gamma_l}{\phi S_w}\right)_{,l}\right]_{,k} = A_{kl} w_l - \frac{b_k^{0f}}{\phi S_w}$$

thermodynamic pressure	capillary pressure	fluid hyper-stress	Darcy dissipation

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thermodynamic pressure	capillary pressure	fluid hyper-stress	Darcy dissipation	bulk forces

Thermodynamical restrictions: generalised Darcy's law

$$\left[-\frac{\partial \Psi_f}{\partial S_w} - \frac{dU}{dS_w} + \left(\frac{\partial \Psi_{nl}}{\partial (\phi S_w)_{,l}}\right)_{,l}\right]_{,k} = A_{kl} w_l - \frac{b_k^{0f}}{\phi S_w}$$

modified capillary energy, $\mathcal{P}_c - \mu$

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modified capillary energy, $\mathcal{P}_c - \mu$

variation of the nonlocal energy

Thermodynamical restrictions: generalised Darcy's law

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generalised chemical potential
(extends the concept of suction)

Thermodynamical restrictions: generalised Darcy's law

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Darcy's law is now a higher order PDE. Admissible bcs are

	w – essential	S_w – essential	μ – natural	τ – natural
w	★		flow & chemical potential	
S_w				
μ			★	
τ				

Thermodynamical restrictions: generalised Darcy's law

$$\left[-\frac{\partial \Psi_f}{\partial S_w} - \frac{dU}{dS_w} + \left(\frac{\partial \Psi_{nl}}{\partial (\phi S_w)_{,l}}\right)_{,l}\right]_{,k} = A_{kl} w_l - \frac{b_k^{0f}}{\phi S_w}$$

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Thermodynamical restrictions: generalised Darcy's law

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τ				

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τ				

Thermodynamical restrictions: generalised Darcy's law

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S_w	■ ■		saturation & chemical potential	saturation & double force
μ	■ ■	■ ■	★	chemical potential & double force
τ	■ ■	■ ■	■ ■	

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S_w	■■		saturation & chemical potential	saturation & double force
μ	■■	■■	★	chemical potential & double force
τ	■■	■■	■■	

where at the boundary: μ-natural bcs

$$\mu = \frac{\partial (\Psi_f + U)}{\partial S_w} - \left(\frac{\partial \Psi_{nl}}{\partial (\phi S_w)_{,l}} \right)_{,l} + \text{terms related to curvature}$$

specify the characteristic of the fluid bath out of the porous medium

Thermodynamical restrictions: generalised Darcy's law

$$\left[-\frac{\partial\Psi_f}{\partial S_w} - \frac{dU}{dS_w} + \left(\frac{\partial\Psi_{nl}}{\partial(\phi S_w)_{,l}}\right)_{,l}\right]_{,k} = A_{kl}w_l - \frac{b_k^{0f}}{\phi S_w}$$

Darcy's law is now a higher order PDE. Admissible bcs are

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μ	■■	■■	★	chemical potential & double force
τ	■■	■■	■■	

where at the boundary: τ-natural bcs

$$\tau_k = \frac{\gamma_l\, n_l}{\phi S_w} n_k = -\left(\frac{\partial\Psi_{nl}}{\partial(\phi S_w)_{,l}} n_l\right) n_k \underset{\text{theory}}{\overset{\text{CH}}{=}} -\kappa\left[(\phi S_w)_{,l} n_l\right] n_k$$

specify the adhesion properties of the fluid to the solid (contact angle, Φ)

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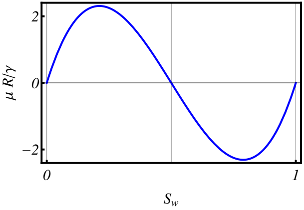
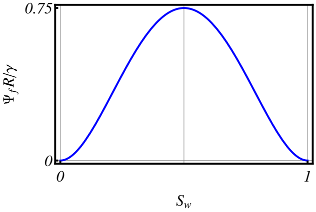
Unsaturated undeformable column (uncoupled problem)

Unsaturated undeformable column

Air-Water biphasic mixture, (CH) energy: $\Psi_f = C\frac{\gamma}{R}S_w^2\left(1 - S_w\right)^2$

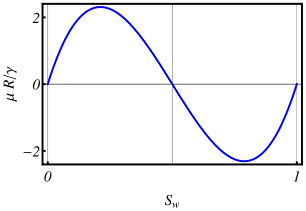
R is the characteristic size corresponding to the transition region between the two phases

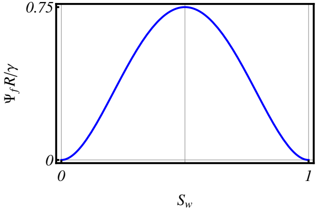




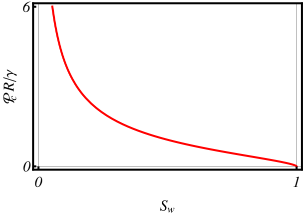
Unsaturated undeformable column

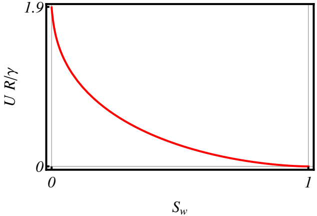
Air-Water biphasic mixture, (CH) energy: $\Psi_f = C\dfrac{\gamma}{R}S_w^2\,(1-S_w)^2$





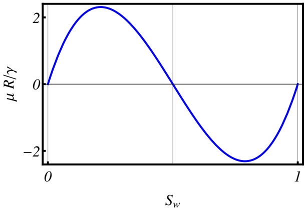
Capillary pressure: $\mathcal{P}_c = \mathtt{C}\left(S_w^{-1/m}-1\right)^{1/n}$ (van Genuchten)

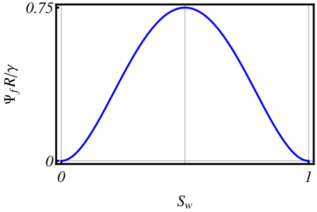




Unsaturated undeformable column

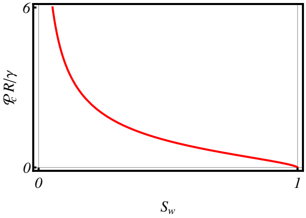
Air-Water biphasic mixture, (CH) energy: $\Psi_f = C\frac{\gamma}{R}S_w^2\,(1 - S_w)^2$

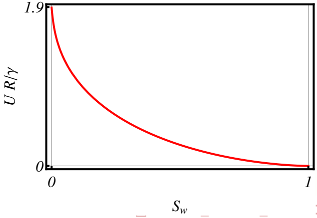




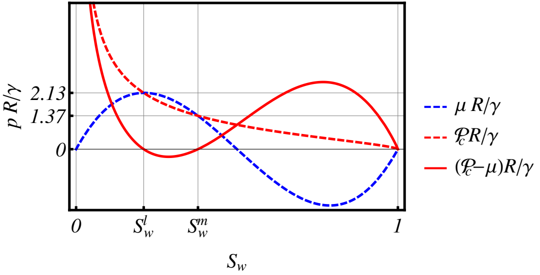
If $\gamma/R = O(\mathsf{C})$ the chemical potential of the bulk fluid, μ, and the capillary pressure, $\mathcal{P}_c$, may be compared (tight reservoir rocks)

Capillary pressure: $\mathcal{P}_c = \mathsf{C}\left(S_w^{-1/m} - 1\right)^{1/n}$ (van Genuchten)

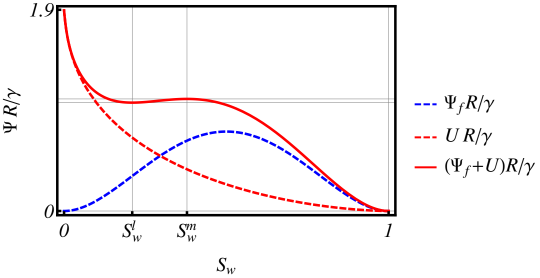




The modified capillary pressure is:



The pore-water free energy is:



Unsaturated undeformable column

Asymptotic analysis in the close vicinity of the external surface

Equilibrium equations

$$\begin{cases} \mu' = g_w, \quad g_w := \rho_w g R^2/\gamma \simeq 0 \\ \mu = \dfrac{\partial\left(\Psi_f + U\right)}{\partial S_w} - \left(\dfrac{\partial \Psi_{nl}}{\partial S'_w}\right)' \end{cases}$$

Contributions to energy

$$\Psi_t \simeq \Psi_f + U, \quad \Psi_{nl} = \tfrac{1}{2}\kappa S_w'^2$$

Boundary conditions

Standard: $\mu = -g_w \simeq 0,\ x/R = 0$

S_w-essential/τ-natural:

τ-natural	$S'_w = 0$	$x/R = 0, \infty$
mixed	$S_w = S^l_w, S'_w = 0$	$x/R = 0, \infty$
S_w-essential	$S_w = S^l_w, S_w = 1$	$x/R = 0, \infty$

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Unsaturated undeformable column

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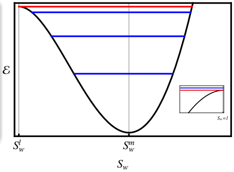
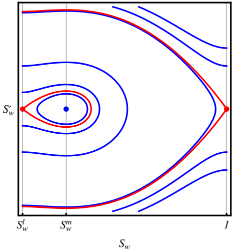
Qualitative analysis

Similarly to 1D mechanical system $x/R \to t$, $' \to \dot{}$

potential energy	kinetic energy
$\mathcal{E} = -(\Psi_f + U)$	$T = \Psi_{nl} = \frac{1}{2}\kappa S'^2_w$





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Unsaturated undeformable column

Asymptotic analysis in the close vicinity of the external surface

Equilibrium equations

$$\begin{cases} \mu' = g_w, \quad g_w := \rho_w g R^2/\gamma \simeq 0 \\ \mu = \dfrac{\partial\left(\Psi_f + U\right)}{\partial S_w} - \left(\dfrac{\partial \Psi_{nl}}{\partial S'_w}\right)' \end{cases}$$

Contributions to energy

$$\Psi_t \simeq \Psi_f + U, \quad \Psi_{nl} = \frac{1}{2}\kappa S_w'^2$$

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τ-natural	$S'_w = 0$	$x/R = 0, \infty$
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Qualitative analysis

Similarly to 1D mechanical system $\quad x/R \to t$, $\; ' \to \dot{}$

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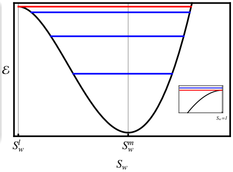

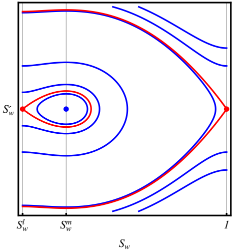

energy level (red) $\Leftrightarrow$ profiles over an unbounded domain

energy level (blue) $\Leftrightarrow$ profiles over a bounded domain & periodic solutions

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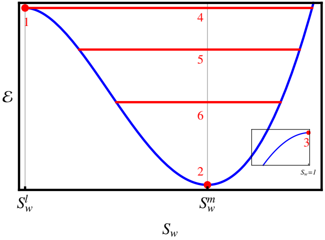
Unsaturated undeformable column

Standard bcs $\mu = -g_w \simeq 0$ in $x/R = 0$

τ-natural bcs $S'_w = 0$ in $x/R = 0, \infty$ $(\Phi = \frac{\pi}{2}$, flat interface limit$)$

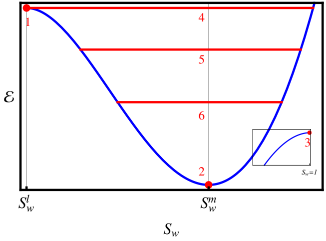


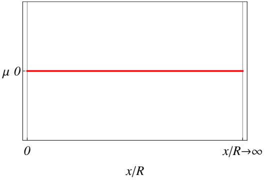
Equilibrium solutions (EqS)

- $1, 2, 3,$ spatially uniform solutions;
- $4,$ "homoclinic" solution;
- $5, 6,$ "periodic" solution;
- etc.

Unsaturated undeformable column

Standard bcs	$\mu = -g_w \simeq 0$	in $x/R = 0$
τ-natural bcs	$S'_w = 0$	in $x/R = 0, \infty$ $\quad (\Phi = \frac{\pi}{2}$, flat interface limit)



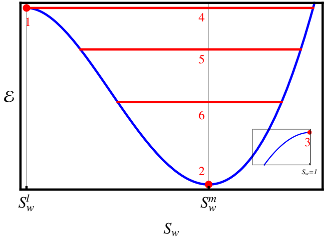


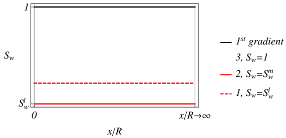
The generalised chemical potential is almost vanishing in the close vicinity of the external surface.

Unsaturated undeformable column

Standard bcs $\mu = -g_w \simeq 0$ in $x/R = 0$

τ-natural bcs $S'_w = 0$ in $x/R = 0, \infty$ $\quad (\Phi = \frac{\pi}{2}$, flat interface limit)





$$\delta^2 \int_{\mathcal{D}} (\Psi_{\text{tot}} + \Psi_{nl}) \Big|_{\text{EqS}_{1,3}} > 0$$

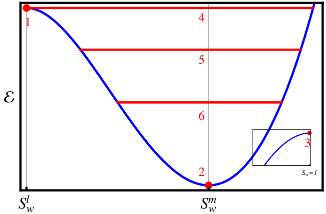
$\text{EqS}_{1,3}$ stable

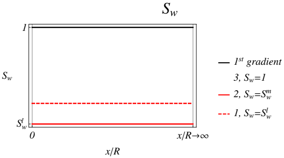
Equilibrium solutions (EqS)

- $1, 2, 3,$ spatially uniform solutions;
- $4,$ "homoclinic" solution;
- $5, 6,$ "periodic" solution;
- etc.

Unsaturated undeformable column

Standard bcs $\mu = -g_w \simeq 0$ in $x/R = 0$
τ-natural bcs $S'_w = 0$ in $x/R = 0, \infty$ ($\Phi = \frac{\pi}{2}$, flat interface limit)





$$\delta^2 \int_{\mathcal{D}} (\Psi_{\text{tot}} + \Psi_{nl}) \Big|_{\text{EqS}_2} < 0$$

EqS$_2$ unstable

Equilibrium solutions (EqS)

- $1, 2, 3,$ spatially uniform solutions;
- $4,$ "homoclinic" solution;
- $5, 6,$ "periodic" solution;
- etc.

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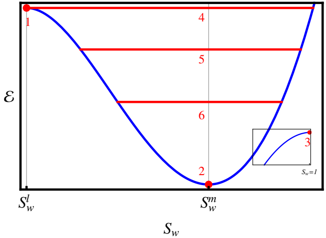
Unsaturated undeformable column

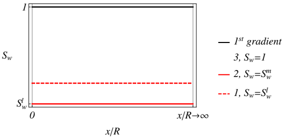
Standard bcs $\mu = -g_w \simeq 0$ in $x/R = 0$

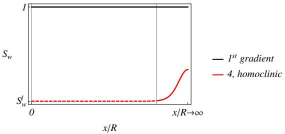
τ-natural bcs $S'_w = 0$ in $x/R = 0, \infty$ ($\Phi = \frac{\pi}{2}$, flat interface limit)



Equilibrium solutions (EqS)

- $1, 2, 3,$ spatially uniform solutions;
- $4,$ "homoclinic" solution;
- $5, 6,$ "periodic" solution;
- etc.





$$\delta^2 \int_{\mathcal{D}} (\Psi_{\text{tot}} + \Psi_{nl}) \Big|_{\text{EqS}_{1,3}} > 0$$

EqS$_{1,3}$ stable

$$\delta^2 \int_{\mathcal{D}} (\Psi_{\text{tot}} + \Psi_{nl}) \Big|_{\text{EqS}_4} < 0$$

EqS$_4$ unstable

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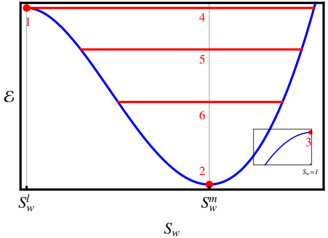
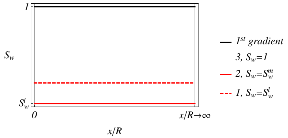
Unsaturated undeformable column

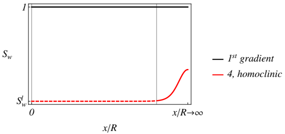
Standard bcs	$\mu = -g_w \simeq 0$	in $x/R = 0$	
τ-natural bcs	$S'_w = 0$	in $x/R = 0, \infty$	$(\Phi = \frac{\pi}{2}$, flat interface limit)

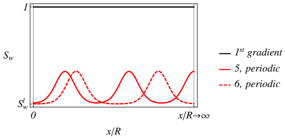


Equilibrium solutions (EqS)

- 1, 2, 3, spatially uniform solutions;
- 4, “homoclinic” solution;
- 5, 6, “periodic” solution;
- etc.







$$\delta^2 \int_{\mathcal{D}} (\Psi_{\rm tot} + \Psi_{nl}) \Big|_{{\rm EqS}_{1,3}} > 0$$

EqS$_{1,3}$ stable

$$\delta^2 \int_{\mathcal{D}} (\Psi_{\rm tot} + \Psi_{nl}) \Big|_{{\rm EqS}_{4}} < 0$$

EqS$_4$ unstable

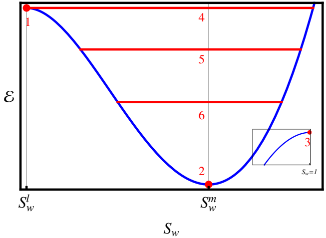
$$\delta^2 \int_{\mathcal{D}} (\Psi_{\rm tot} + \Psi_{nl}) \Big|_{{\rm EqS}_{5,6}} < 0$$

EqS$_{5,6}$ unstable

Unsaturated undeformable column

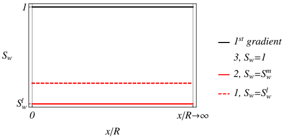
Standard bcs $\mu = -g_w \simeq 0$ in $x/R = 0$

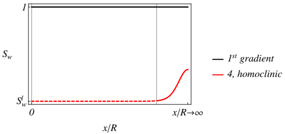
τ-natural bcs $S'_w = 0$ in $x/R = 0, \infty$ ($\Phi = \frac{\pi}{2}$, flat interface limit)

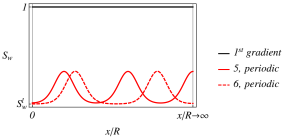


Equilibrium solutions (EqS)

- 1, 2, 3, spatially uniform solutions;
- 4, "homoclinic" solution;
- 5, 6, "periodic" solution;
- etc.







$$\delta^2 \int_{\mathcal{D}} (\Psi_{\rm tot} + \Psi_{nl}) \Big|_{\text{EqS}_{1,3}} > 0$$

EqS$_{1,3}$ stable

(EqS$_3$ absolute minimum)

$$\delta^2 \int_{\mathcal{D}} (\Psi_{\rm tot} + \Psi_{nl}) \Big|_{\text{EqS}_{4}} < 0$$

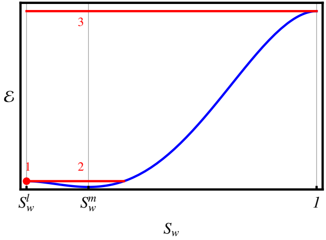
EqS$_4$ unstable

$$\delta^2 \int_{\mathcal{D}} (\Psi_{\rm tot} + \Psi_{nl}) \Big|_{\text{EqS}_{5,6}} < 0$$

EqS$_{5,6}$ unstable

Unsaturated undeformable column (uncoupled problem)

Standard bcs $\mu = -g_w \simeq 0$ in $x/R = 0$
Mixed bcs $S_w = S_w^l,\ S_w' = 0$ in $x/R = 0, \infty$

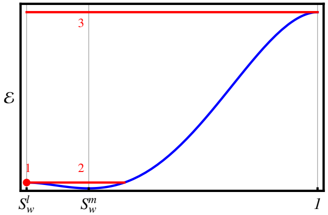


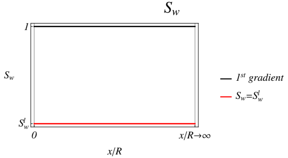
Equilibrium solutions (EqS)

- 1, spatially uniform solution;
- 2, "homoclinic profile" with a vanishing slope at $x/R = 0$ and $x/R \to \infty$
- 3, "profile" with a vanishing slope at $S_w = 1$

Unsaturated undeformable column (uncoupled problem)

Standard bcs $\mu = -g_w \simeq 0$ in $x/R = 0$
Mixed bcs $S_w = S_w^l,\ S_w' = 0$ in $x/R = 0, \infty$





$$\delta^2 \int_{\mathcal{D}} (\Psi_{\text{tot}} + \Psi_{nl}) \Big|_{\text{EqS}_1} > 0$$

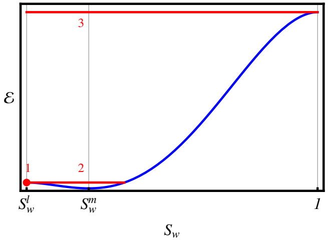
EqS$_1$ stable

Equilibrium solutions (EqS)

- 1, spatially uniform solution;
- 2, "homoclinic profile" with a vanishing slope at $x/R = 0$ and $x/R \to \infty$
- 3, "profile" with a vanishing slope at $S_w = 1$

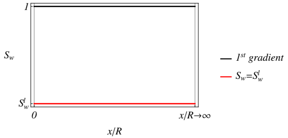
Unsaturated undeformable column (uncoupled problem)

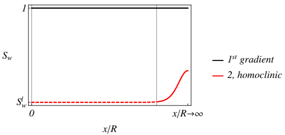
Standard bcs		$\mu = -g_w \simeq 0$	in $x/R = 0$
Mixed	bcs	$S_w = S_w^l,\ S_w' = 0$	in $x/R = 0, \infty$



Equilibrium solutions (EqS)

- 1, spatially uniform solution;
- 2, "homoclinic profile" with a vanishing slope at $x/R = 0$ and $x/R \to \infty$
- 3, "profile" with a vanishing slope at $S_w = 1$





$$\delta^2 \int_{\mathcal{D}} (\Psi_{\text{tot}} + \Psi_{nl}) \Big|_{\text{EqS}_1} > 0$$

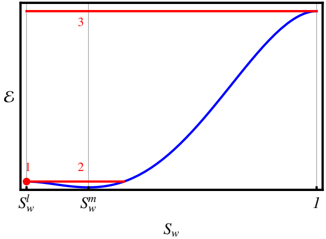
EqS$_1$ stable

$$\delta^2 \int_{\mathcal{D}} (\Psi_{\text{tot}} + \Psi_{nl}) \Big|_{\text{EqS}_2} < 0$$

EqS$_2$ unstable

Standard bcs $\mu = -g_w \simeq 0$ in $x/R = 0$

Mixed bcs $S_w = S_w^l,\ S_w' = 0$ in $x/R = 0, \infty$



Equilibrium solutions (EqS)

- 1, spatially uniform solution;
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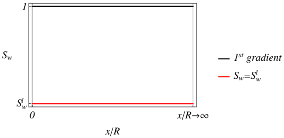
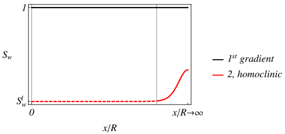
- 3, "profile" with a vanishing slope at $S_w = 1$



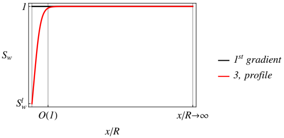
$$\delta^2 \int_{\mathcal{D}} (\Psi_{\text{tot}} + \Psi_{nl}) \Big|_{\text{EqS}_1} > 0$$

EqS$_1$ stable



$$\delta^2 \int_{\mathcal{D}} (\Psi_{\text{tot}} + \Psi_{nl}) \Big|_{\text{EqS}_2} < 0$$

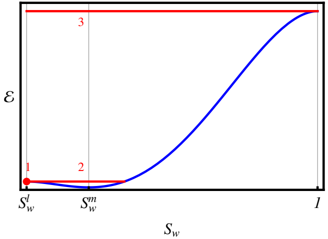
EqS$_2$ unstable



$$\delta^2 \int_{\mathcal{D}} (\Psi_{\text{tot}} + \Psi_{nl}) \Big|_{\text{EqS}_3} > 0$$

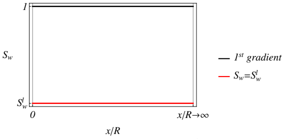
EqS$_3$ stable

Standard	bcs	$\mu = -g_w \simeq 0$	in $x/R = 0$
Mixed	bcs	$S_w = S_w^l,\ S_w' = 0$	in $x/R = 0, \infty$



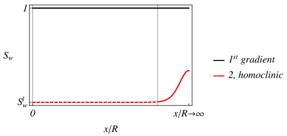
Equilibrium solutions (EqS)

- 1, spatially uniform solution;
- 2, "homoclinic profile" with a vanishing slope at $x/R = 0$ and $x/R \to \infty$
- 3, "profile" with a vanishing slope at $S_w = 1$



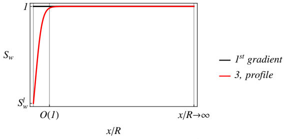
$$\delta^2 \int_{\mathcal{D}} (\Psi_{\rm tot} + \Psi_{nl}) \Big|_{\rm EqS_1} > 0$$

EqS$_1$ stable



$$\delta^2 \int_{\mathcal{D}} (\Psi_{\rm tot} + \Psi_{nl}) \Big|_{\rm EqS_2} < 0$$

EqS$_2$ unstable



$$\delta^2 \int_{\mathcal{D}} (\Psi_{\rm tot} + \Psi_{nl}) \Big|_{\rm EqS_3} > 0$$

EqS$_3$ stable

(EqS$_3$ absolute minimum)

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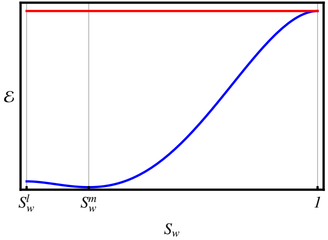
Unsaturated undeformable column

Standard	bcs	$\mu = -g_w \simeq 0$	in $x/R = 0$
S_w Essential	bcs	$S_w = S_w^l,\ S_w = 1$	in $x/R = 0, \infty$



Equilibrium solutions (EqS)

– "profile" with a vanishing slope at $S_w = 1$

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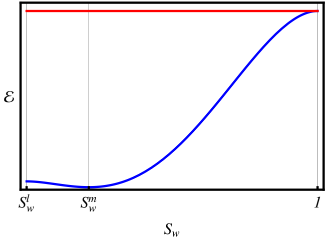

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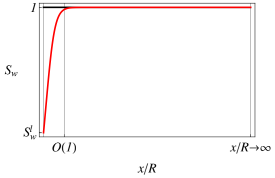
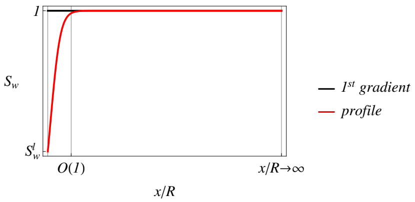


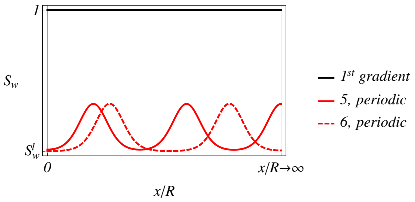
$$\delta^2 \int_{\mathcal{D}} (\Psi_{tot} + \Psi_{nl}) \Big|_{EqS} > 0$$

EqS stable

Unsaturated undeformable column

... Novelties & further developments ...





The absolute minimizer of energy is a profile with a boundary layer in the neighborhood of the external surface

Periodic solution are also admissible even if unstable.

Proposition

Modifying the CH model into a more sophisticated one, say $\kappa \to \kappa(S_w)$, may provide stable solutions describing a (quasi-) discontinuous fluid distribution within the column.

Conclusions & further developments

- A thermodynamically consistent phase-field model of unsaturated poromechanics, based on gradient theory, has been stated.
- Partial saturation has been accounted for by means of a biphasic fluid which saturates the pore space.
- A pore-scale uncoupled analysis has been developed for showing the characteristic feature of the model.

Conclusions & further developments

- A thermodynamically consistent phase-field model of unsaturated poromechanics, based on gradient theory, has been stated.
- Partial saturation has been accounted for by means of a biphasic fluid which saturates the pore space.
- A pore-scale uncoupled analysis has been developed for showing the characteristic feature of the model.

- A complete implementation of the model for a deformable skeleton must be developed.
- Plasticization and damage of the skeleton must be introduced into the model with the aim of capturing localisation induced by wetting & drying.
- More than a single biphasic fluid must be accounted for, so as to model fluid co-capillarity and fluid trapping.