on (a,b,c)

a. 'thermodynamic state'

b. 'critical state',

c. 'rate and state'

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thermodynamic state (pure solids)

Helmholtz free energy $f = f\left(\epsilon_{ij}^{e}\right)$

Rates of internal variables $\dot{\epsilon}_{ij}^{e}\left(\epsilon_{ij}^{e}, \dot{\epsilon}_{ij}\right) = \dot{\epsilon}_{ij} - \dot{\epsilon}_{ij}^{p}\left(\epsilon_{ij}^{e}, \dot{\epsilon}_{ij}\right)$

1) Steady state $\dot{\epsilon}_{ij}^{e}\left(\epsilon_{ij}^{e}, \dot{\epsilon}_{ij}\right) \equiv 0$

Stress $\sigma_{ij}\left(\epsilon_{ij}^{e}\right) = \partial f / \partial \epsilon_{ij}^{e}$ → 2) $\epsilon_{ij}^{e} \equiv \epsilon_{ij}^{e}\left(\sigma_{ij}\right)$

3) From equations 1 and 2 $\dot{\epsilon}_{ij}^{e}\left(\sigma_{ij}, \dot{\epsilon}_{ij}\right) \equiv 0$

4) Rate independence $\dot{\epsilon}_{ij}^{e}\left(\sigma_{ij}, \lambda\dot{\epsilon}_{ij}\right) = \lambda\dot{\epsilon}_{ij}^{e}\left(\sigma_{ij}, \dot{\epsilon}_{ij}\right) \equiv 0$

or $\dot{\epsilon}_{ij}^{e}\left(\sigma_{ij}\right) \equiv 0$

failure state (pure solids)

Summary. Under the following three assumptions:

(1) energy dependent only on elastic strain;

(2) rate independence; and

(3) steady state,

we find $\dot{\epsilon}^e_{ij}\left(\sigma_{ij}\right) \equiv 0$

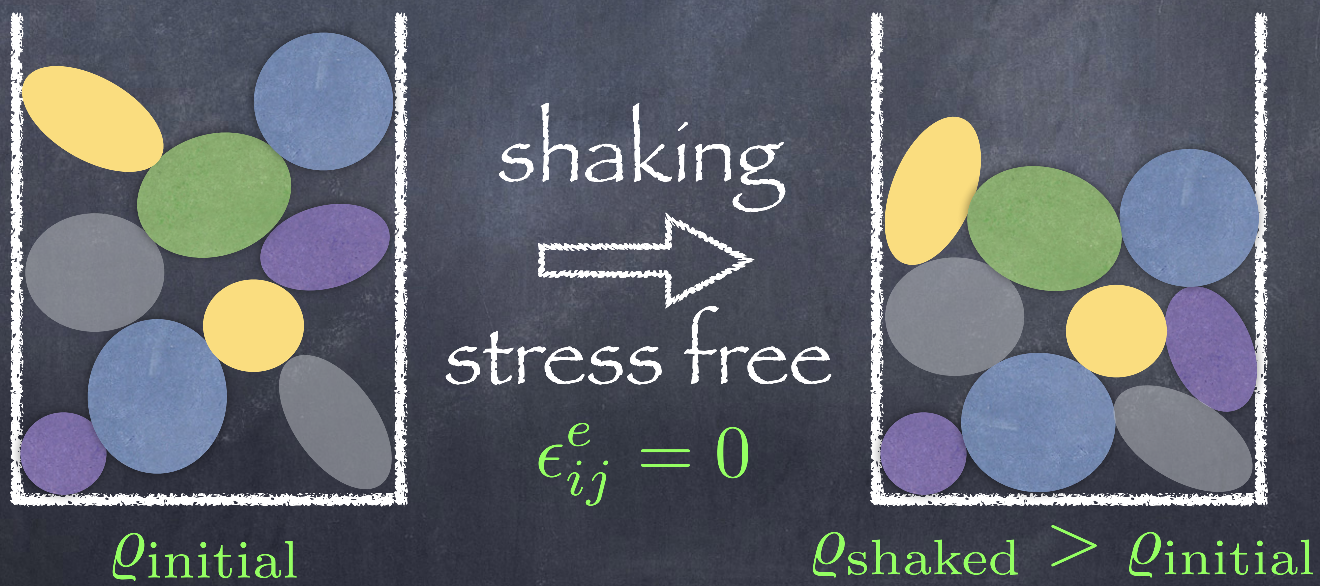
'failure criterion' $\boxed{F\left(\sigma_{ij}\right) \equiv 0}$

thermodynamic state (general soils)

Helmholtz free energy

$$f \equiv f\left(\varrho, \epsilon^e_{ij}, \ldots\right)$$

Landau (1953)
de Gennes (1993)
Jiang-Liu (2009)



thermodynamic state ('classical' soils)

Helmholtz free energy $f = f\left(\varrho, \epsilon^e_{ij}\right)$

Rates of internal variables $\dot{\varrho} = \varrho\dot{\epsilon}_v$

$$\dot{\epsilon}^e_{ij} = \dot{\epsilon}_{ij} - \dot{\epsilon}^p_{ij}\left(\varrho, \epsilon^e_{ij}, \dot{\epsilon}_{ij}\right)$$

1) Steady state $\dot{\varrho} = \varrho\dot{\epsilon}_v \equiv 0$

$$\dot{\epsilon}^e_{ij} = \dot{\epsilon}_{ij} - \dot{\epsilon}^p_{ij}\left(\varrho, \epsilon^e_{ij}, \dot{\epsilon}_{ij}\right)$$

$$= \dot{\epsilon}^e_{ij}\left(\varrho, \epsilon^e_{ij}, \dot{\epsilon}_{ij}\right) \equiv 0$$

Landau (1953)
de Gennes (1993)
Jiang-Liu (2009)

2) Stress $\sigma_{ij} = \partial f / \partial \epsilon^e_{ij} + P_T \delta_{ij}$

$$P_T\left(\varrho, \epsilon^e_{ij}\right) = -\frac{\partial(f/\varrho)}{\partial(1/\varrho)}$$

➡ $\epsilon^e_{ij} \equiv \epsilon^e_{ij}\left(\varrho, \sigma_{ij}\right)$

thermodynamic state ('classical' soils)

3) From equations 1 and 2 $\dot{\epsilon}^e_{ij}\left(\varrho, \sigma_{ij}, \dot{\epsilon}_{ij}\right) = 0$

since $\dot{\epsilon}_v = 0$ → $\dot{\epsilon}^e_{ij}\left(\varrho, \sigma_{ij}, \dot{\epsilon}_s\right) = 0$

4) Rate independence $\dot{\epsilon}^e_{ij}\left(\varrho, \sigma_{ij}, \lambda\dot{\epsilon}_s\right) = \lambda\dot{\epsilon}^e_{ij}\left(\varrho, \sigma_{ij}, \dot{\epsilon}_s\right) \equiv 0$

critical state ('classical' soils)

Summary. Under the following three assumptions:

(1) energy dependent on elastic strain and density;

(2) rate independence; and

(3) steady state,

we find

critical state 'line' $\boxed{F\left(\varrho, \sigma_{ij}\right) \equiv 0}$

$$e = \frac{\hat{\varrho} - \varrho}{\varrho}$$

critical state (the ‘α’ soil)

Summary. Under the following three assumptions:

(1) energy dependent on elastic strain and density and a vector of internal variables α_{ij};

(2) rate independence; and

(3) steady state,

critical state ‘field’ $F\left(\varrho, \sigma_{ij}, \alpha_{ij}\right) \equiv 0$

$$e = \frac{\hat{\varrho} - \varrho}{\varrho}$$

critical state ('crushable' soils)

Summary. Under the following three assumptions:

(1) energy dependent on elastic strain and density and breakage B;

(2) rate independence; and

(3) steady state,

↘

critical state 'field' $F\left(\varrho, \sigma_{ij}, B\right) \equiv 0$

critical state ('crushable' soils)

$$E_B\left(\varrho, \sigma_{ij}\right) = \frac{\partial f}{\partial B} \qquad \tau\left(\varrho, B\right) = \frac{\varrho_{\max}(B) - \varrho}{\varrho_{\max}(B) - \varrho_{\min}(B)}$$

Tengattini et al (2016)

$$F\left(\varrho, \sigma_{ij}, B\right) = \sqrt{\frac{E_B}{E_c}} - \gamma\tau \equiv 0$$

critical state ('crushable' soils)

$$E_B\left(\varrho, \sigma_{ij}\right) = \frac{\partial f}{\partial B} \qquad \tau\left(\varrho, B\right) = \frac{\varrho_{\max}(B) - \varrho}{\varrho_{\max}(B) - \varrho_{\min}(B)}$$

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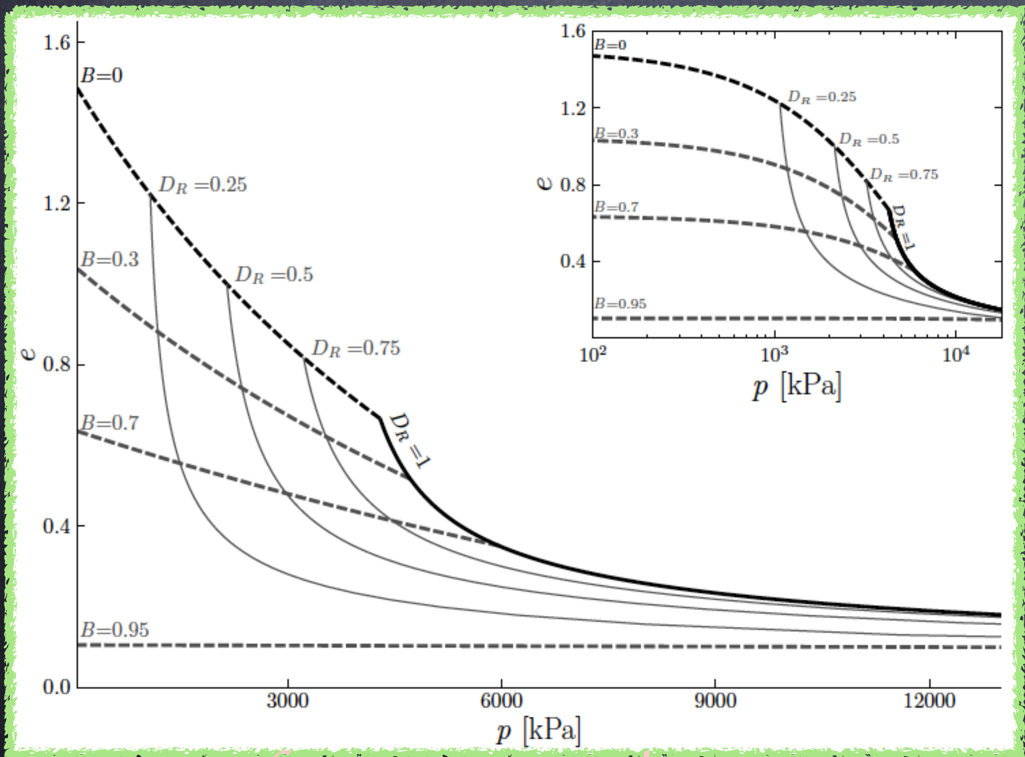
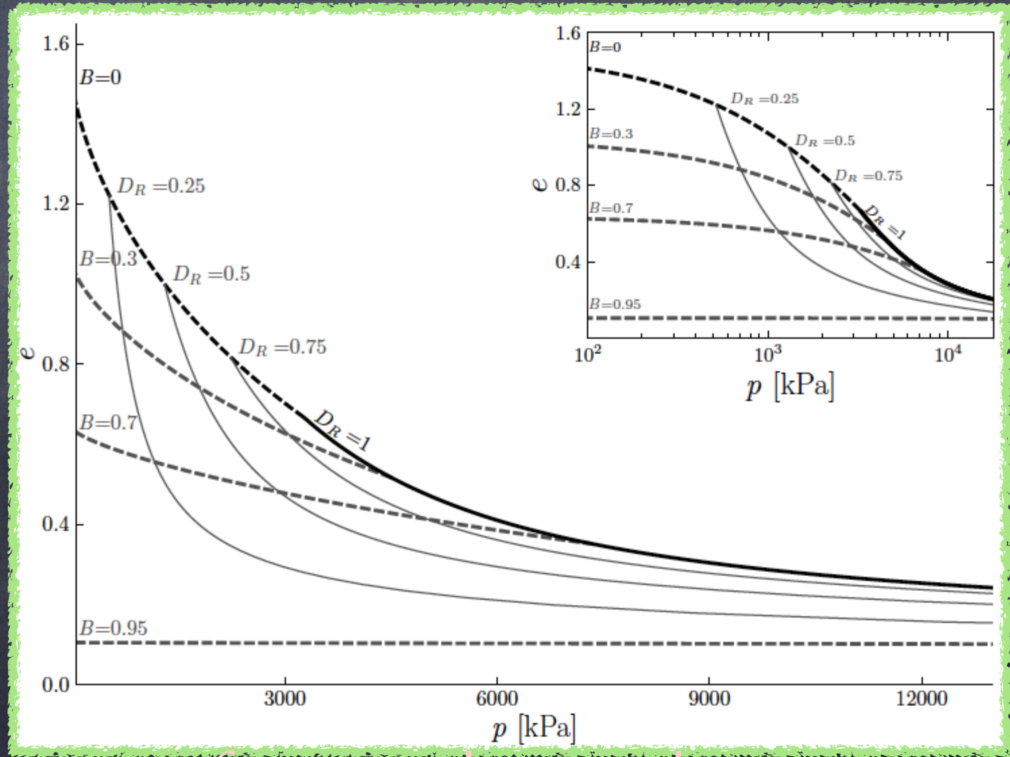
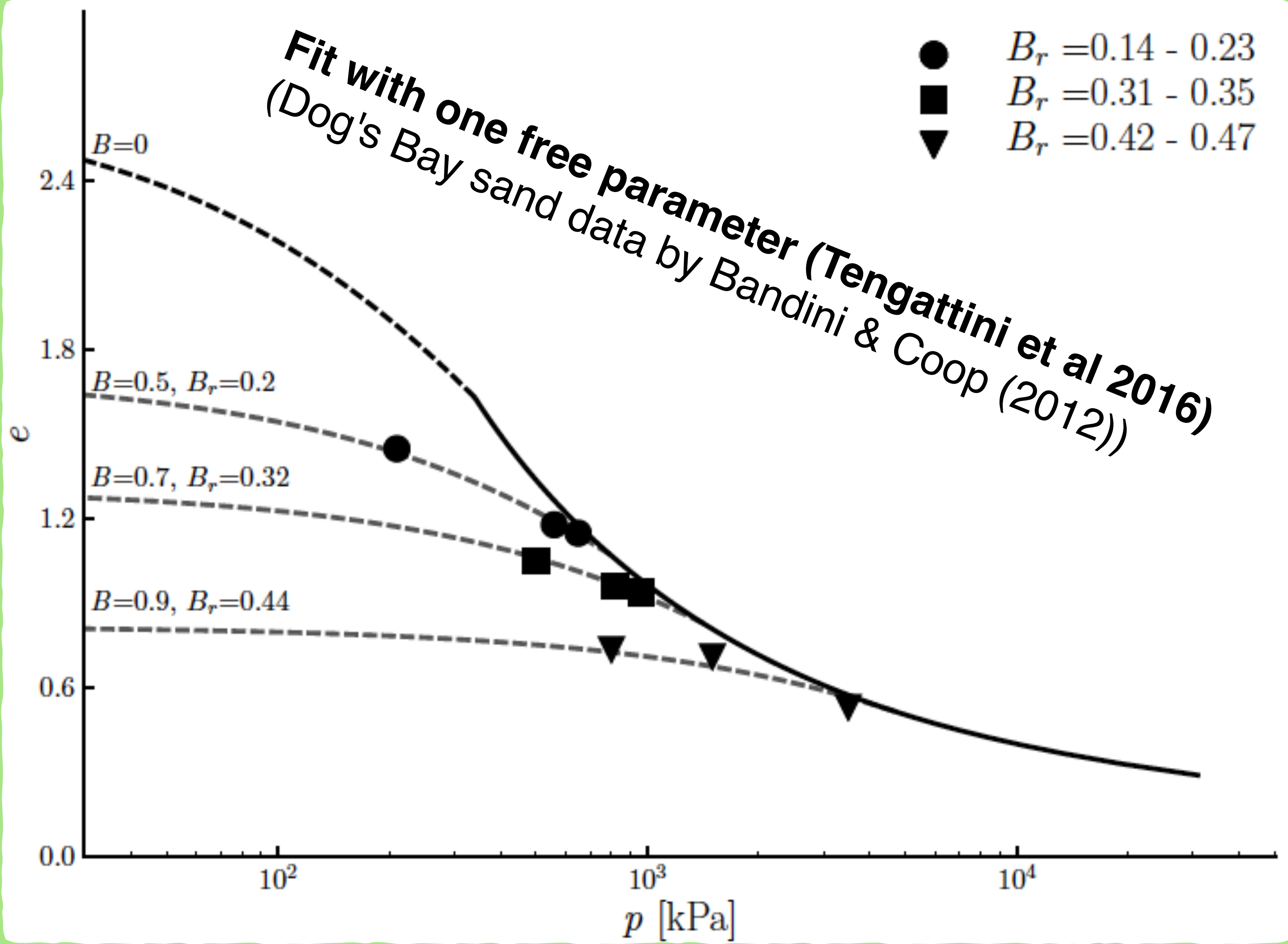


E_B from Linear Elasticity



E_B from Nonlinear Elasticity

critical state ('crushable' soils)



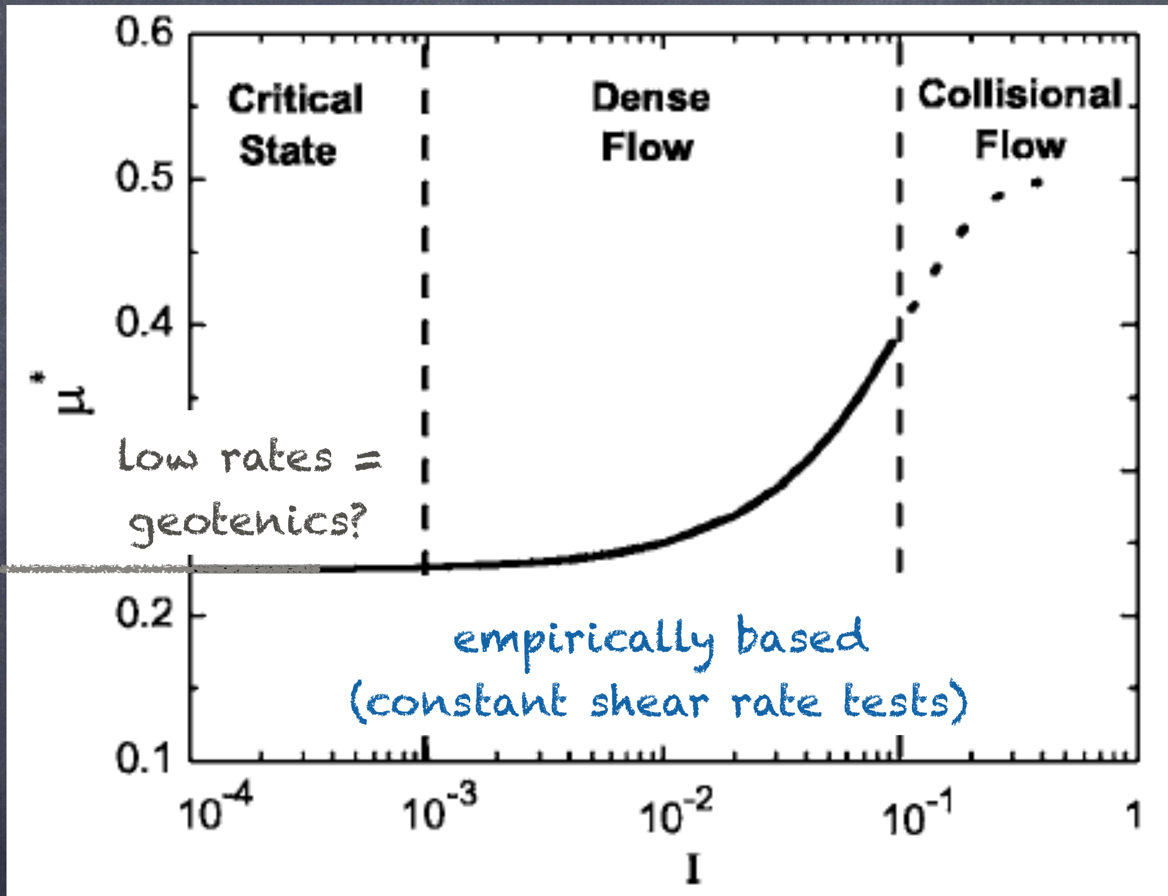
sub-conclusions

- ‘Critical State Theory’ (CST) builds on the assumption that steady state will eventuate at large strains under very low shear rates

- critical state is necessarily dependent on the form of thermodynamic energy

μ(I)-rheology

da Cruz et al (2011)



$$I \equiv d\dot{\gamma}\sqrt{\frac{\varrho}{\sigma_{ii}}}$$

low rates = geotenics?

$$F\left(\varrho, \sigma_{ij}\right) \equiv 0$$

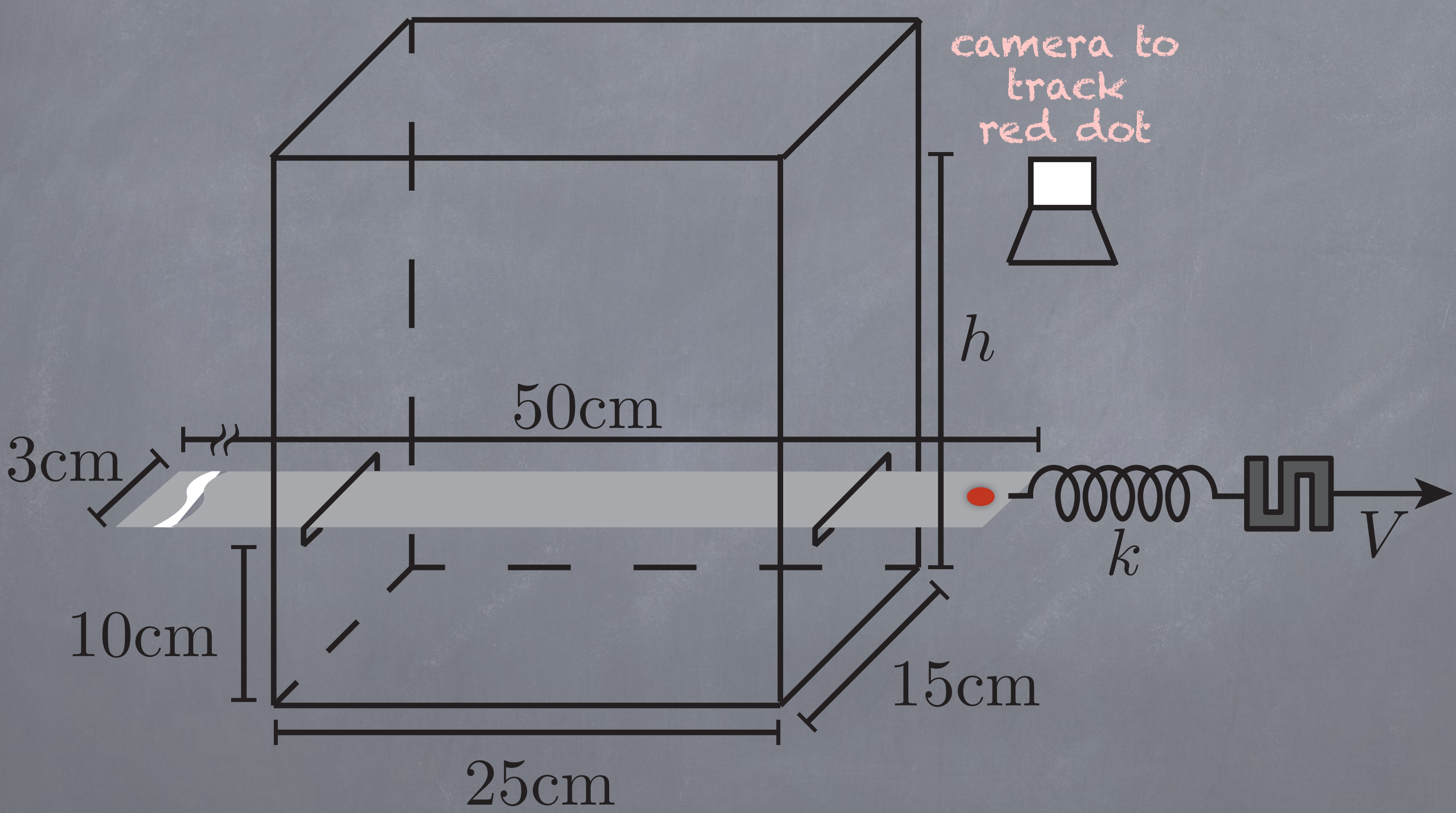
empirically based
(constant shear rate tests)

$$F\left(\varrho, \sigma_{ij}, \dot{\epsilon}_s\right) \equiv 0$$

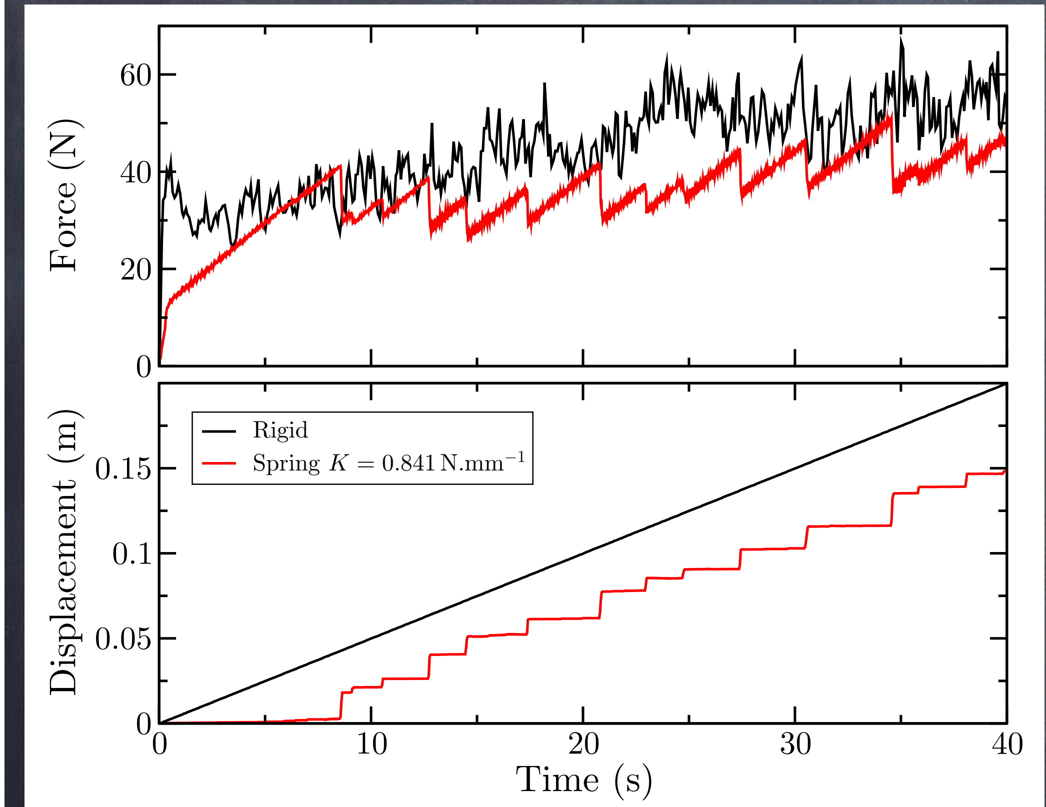
$$\tau_{xy} = \sigma_{yy}\mu(I)$$

$$\varrho = \varrho(I)$$

non-critical state (stick-slip)

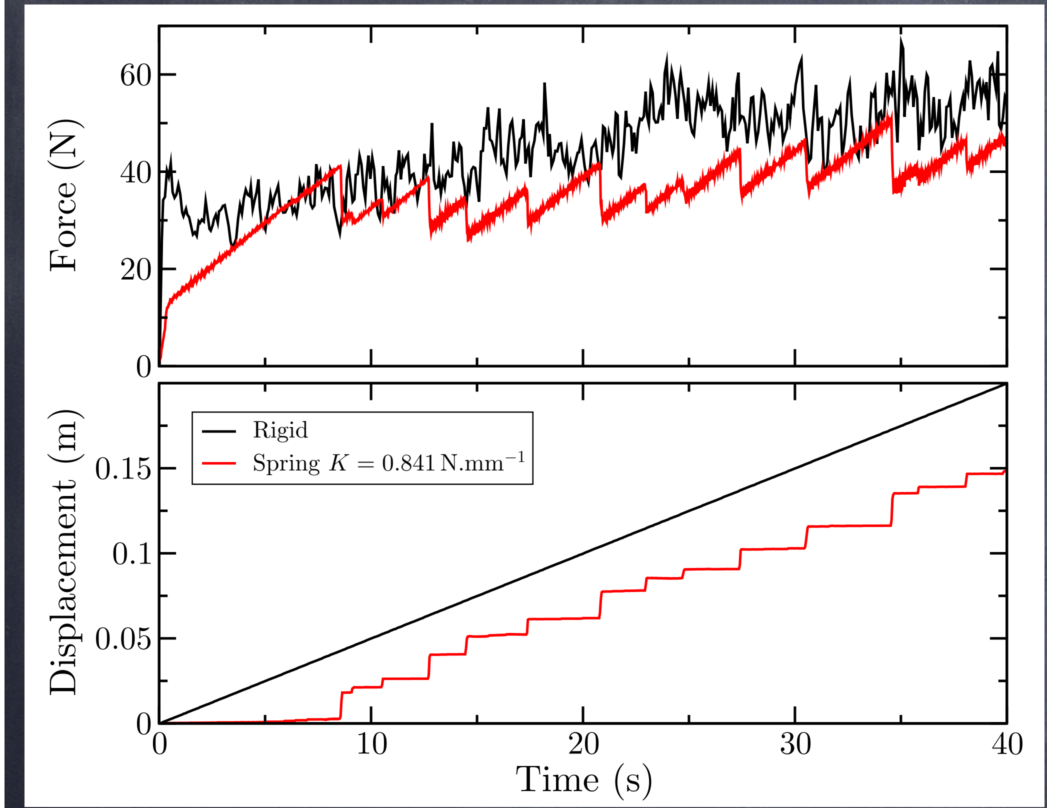


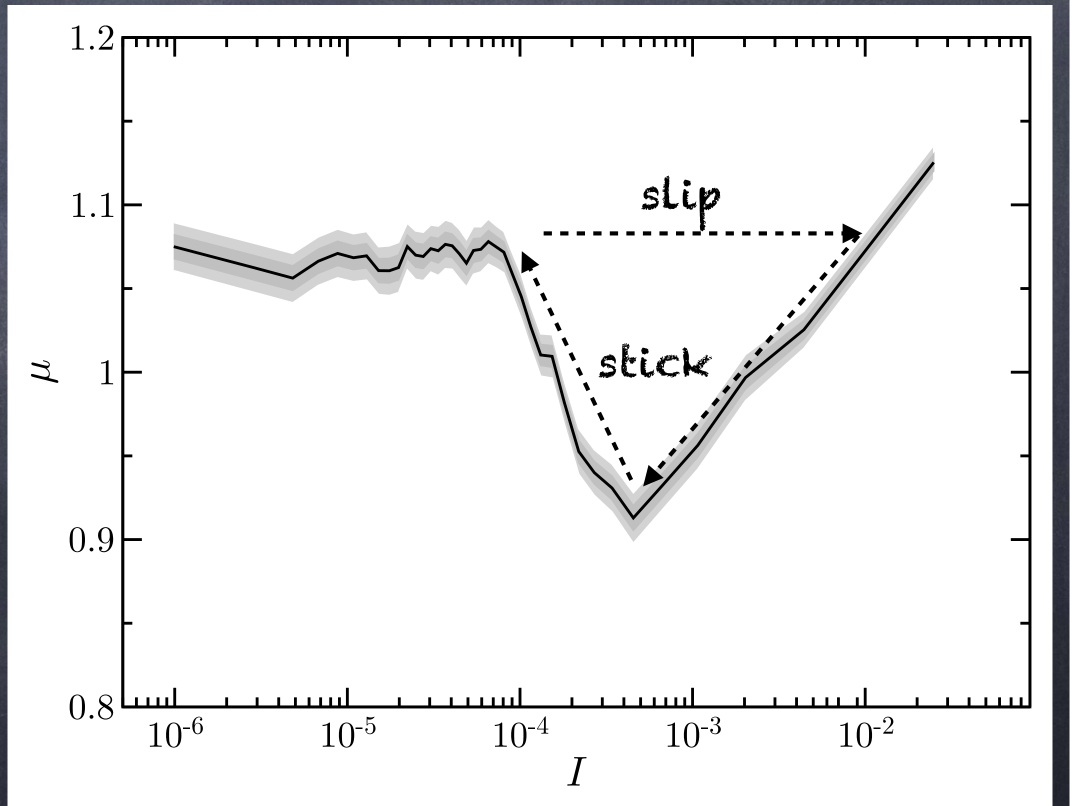
non-critical state (stick-slip)



update μ(I) ?

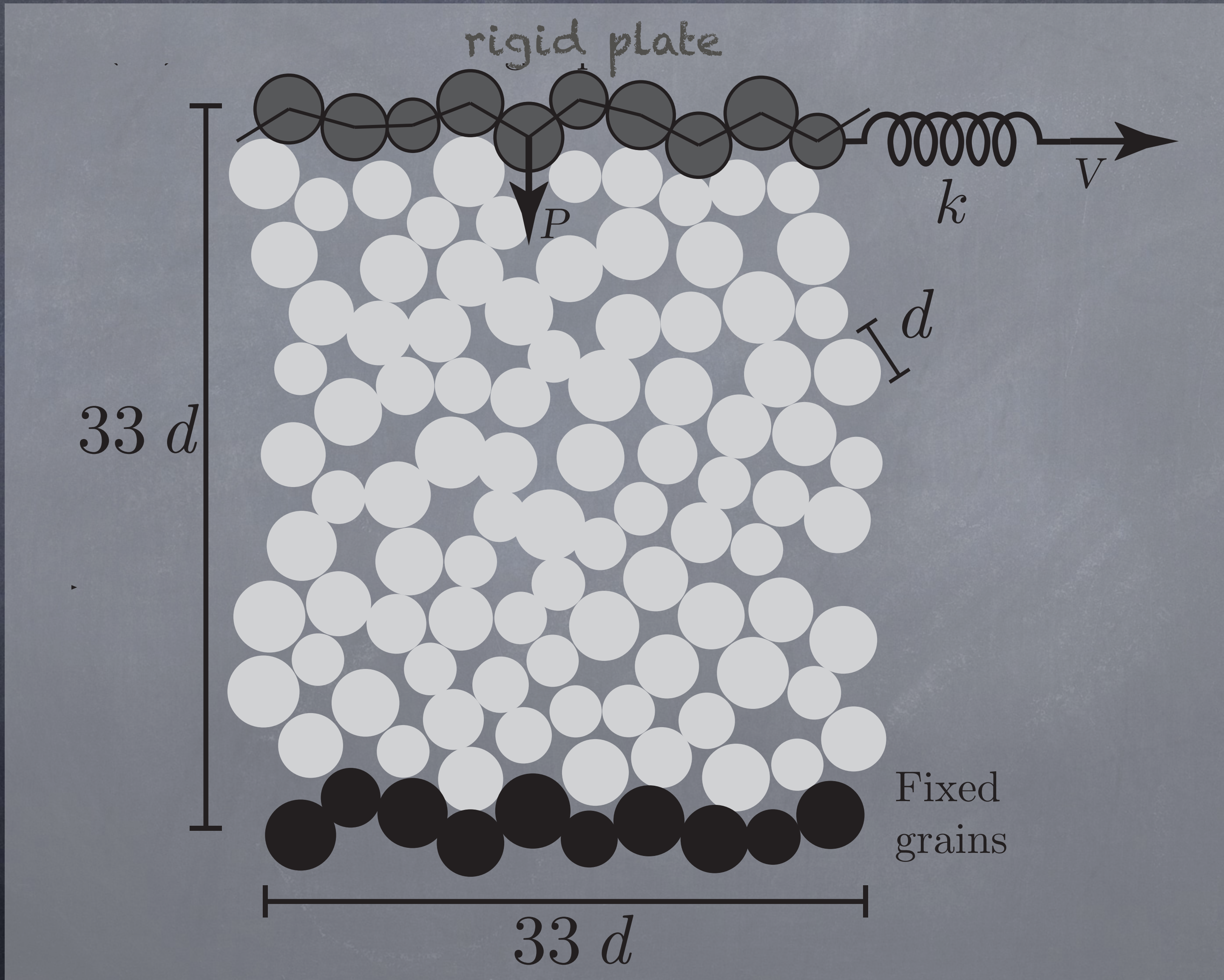
non-critical state (stick-slip)



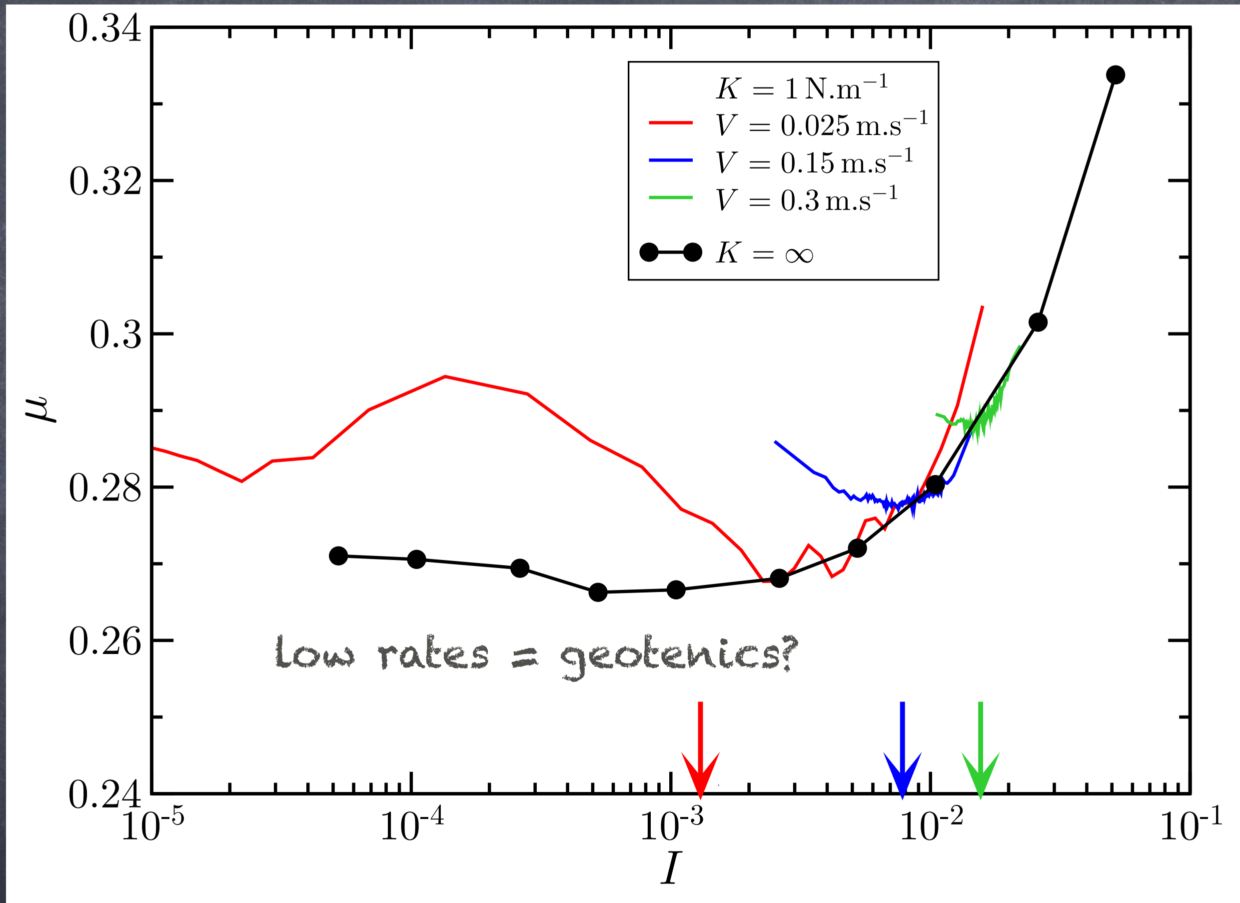


update μ(I) ?

non-critical state (stick-slip)



non-critical state (stick-slip)



- what will CST predict? → constant force
- update μ(I) ? → but no uniqueness

non-CST
(rate and state friction law)

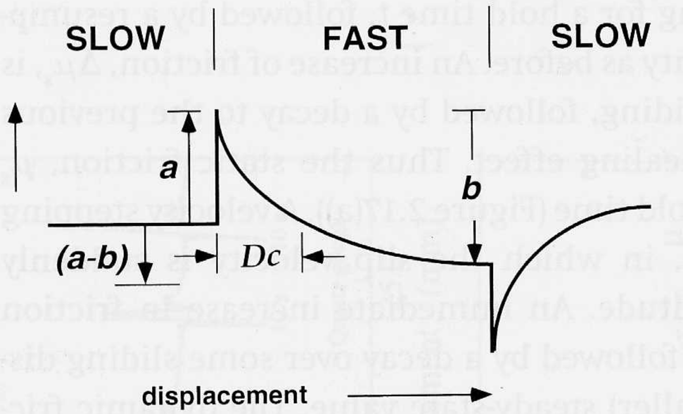


Fig. 2.18. Schematic diagram showing the response to *e*-fold velocity changes and defining the terms in the rate-state friction law.

(Scholz, 1998)

$$\frac{\tau}{\sigma} \equiv \mu = \mu_0 + a \ln\left(\frac{V}{V_0}\right) + b \ln\left(\frac{V_0\theta}{L}\right)$$

$$\dot{\theta} = 1 - \theta\frac{V}{L}$$

Dieterich (1981)
Ruina (1983)

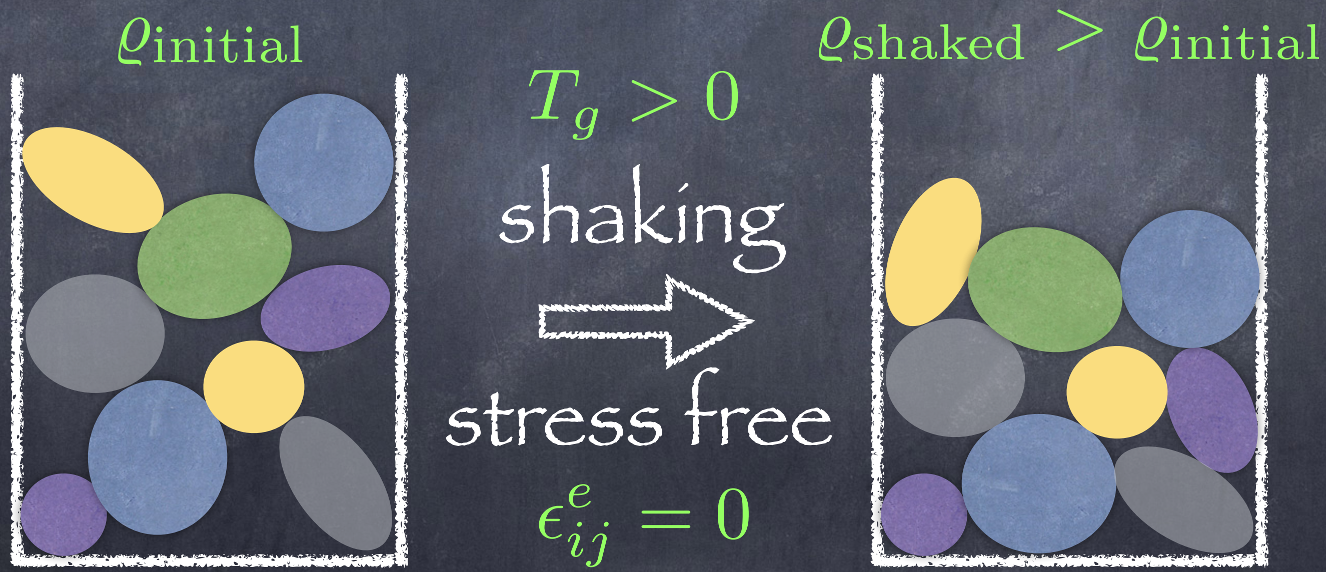
The critical slip distance L is interpreted as the sliding distance required to renew contact population. In this view θ represents average contact lifetime.

thermodynamic state (granular soils)

Helmholtz free energy

$$f \equiv f\left(\varrho, \epsilon_{ij}^{e}, T_g, \ldots\right)$$

Landau (1953)
de Gennes (1993)
Jiang-Liu (2009)



Granular Solids Hydrodynamics

Helmholtz free energy $f = f\left(\varrho, \epsilon^e_{ij}, T_g\right)$

Jiang-Liu (2009)

Stress $\sigma_{ij} = \partial f / \partial \epsilon^e_{ij} + P_T \delta_{ij} - \sigma^D_{ij}$

$$P_T\left(\varrho, \epsilon^e_{ij}, T_g\right) = -\frac{\partial(u/\varrho)}{\partial(1/\varrho)} \propto T_g^2$$

$$\sigma^D_{ij} \propto \dot{\epsilon}_s^2$$

$$\mu = \frac{\sigma_s}{\sigma_v} = \frac{\partial f/\partial \epsilon^e_s - c_1 \dot{\epsilon}_s^2}{\partial f/\partial \epsilon^e_v + c_2 T_g^2}$$

i.e., μ = μ(density, stress, 'rate and state')

conclusions

- CST builds on the assumption that steady state will eventuate under very low shear rates
- which violates transient stick-slip observations
- adding granular temperature to describe thermodynamic state can resolve this issue