Critical states and critical state theory

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It would be a pity to insist on a revision to the definition of critical states for what, to me, appear to be historically unnecessary reasons. The book *Critical state soil mechanics* is nearing its 50th anniversary and, does indeed use the phrase *critical state theory* from time to time. However, I would contend that this particular phrase has not been generally and consistently adopted. If it is to be used then it needs to be clearly and unambiguously defined.

Two things are important about the definition of a critical state implied by Schofield and Wroth. Firstly, they restricted themselves (almost exclusively) to triaxial compression so that there was no need for them to repeat any statement referring particularly to triaxial compression; and secondly they stated frequently that a critical state necessarily implied not only that the stresses and density should have stopped changing but also that the soil should be being continually sheared with increasing distortional strain $\dot{\epsilon}_a > \dot{\epsilon}_r$. Thus this critical state can be written

$$\frac{\partial p'}{\partial \epsilon_q} = \frac{\partial q}{\partial \epsilon_q} = \frac{\partial v}{\partial \epsilon_q} = 0 \tag{1}$$

or, daringly

$$\dot{p}' = \dot{q} = \dot{v} = 0 \quad \text{and} \quad \dot{\epsilon}_q > 0 \tag{2}$$

Here p', q and v are mean effective stress, distortional stress, and specific volume and ϵ_q is distortional strain. These days we might add statements about grading, particle shape, and fabric having also reached stationary values - on average - but that does not upset the original definition.

If the direction of shearing is reversed then we are certainly breaking the critical state requirements and changes in effective stress and in volume (not necessarily monotonic) will occur until once again an appropriate combination of stresses and specific volume is attained - in triaxial extension

$$\dot{p}' = \dot{q} = \dot{v} = 0 \quad \text{and} \quad \dot{\epsilon}_q < 0 \tag{3}$$

but we are now somewhat on our own and it is not certain that these values of p', $|q|$, v will be identical to those previously reached in triaxial compression. We might make the constitutive assumption that they should be the same but it is not obvious that this must be so.

It may be instructive to use the plane strain deviatoric section of stress space to present some more general results typical of tests with changes in the direction of the applied distortional strains. The section has axes $\beta = (\sigma_z - \sigma_x)/2$ and τ_{zx}. One can imagine the out-of-plane stress σ_y being a principal stress We can identify stress states corresponding to the triaxial compression and triaxial section. Corresponding work conjugate strain increments are shear strain $\delta\gamma_{zx}$; and in-plane volumetric strain $\delta\epsilon_s = \delta\epsilon_x + \delta\epsilon_z/2$.

An important consequence of this choice of stress increment quantities is that the radial distance from the origin of the deviatoric stress is the in-plane principal stress $\sigma_1 - \sigma_3$ and the angle made by this stress vector with the horizontal

β axis is the angle made by the major principal stress σ_1 with the horixontal. In addition, the Mohr-Coulomb failure criterion plots as a circle centred on the origin.

When we reach a critical state in this stress plane, the strain increments and stresses are more or less coaxial (Roscoe, Bassett and Cole, 1967), (Ibraim et al, 2010?). Thus the imposition of a constant strain increment rate implies a correspondingly aligned eventual critical state stress. Change of strain rate direction will imply a corresponding change in critical state stress. A circular strain path implies a constantly changing direction of shearing so that the requirement for continued shearing is immediately breached. The stress state, meanwhile, will be constantly hunting for the point on the failure circle matching the direction of principal strain which is imposed *at that instant* only.

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