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Effect of constitutive choices on the energetics: selected examples

Eleni Gerolymatou

Geotechnical Research Group

Introduction

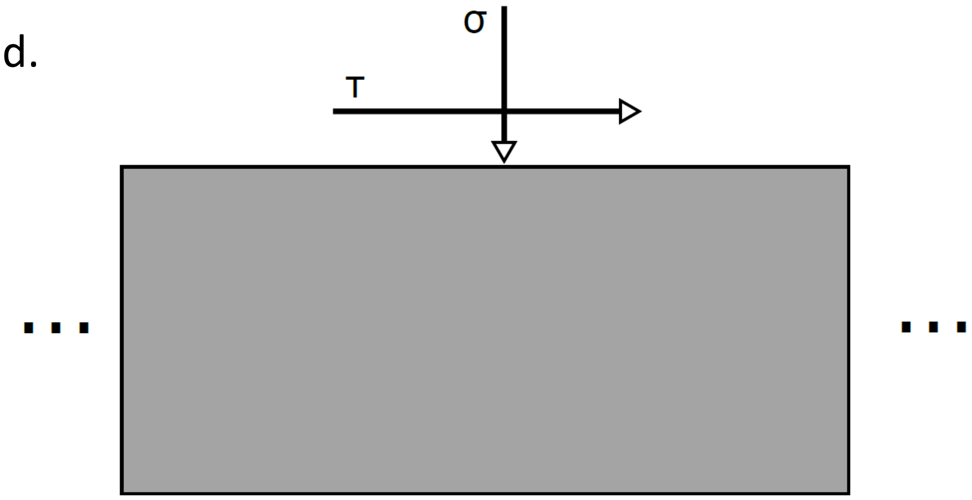
- Shear of an infinite strip
- Anisotropic elasticity
- Elastoplasticity
- Thermal diffusion

Shear of an infinite strip - Cauchy

- An infinite strip of material is considered.
- The internal virtual power reads

$$P_{int} = \sigma_{yy} u_{y,y} + \sigma_{xy} u_{x,y}$$

- The height is 2 m
- The applied shear stress is 0.5 Pa.
- The applied normal stress is 0.25 Pa.
- The elastic modulus is 1000 Pa.
- The Poisson ratio is 0.0

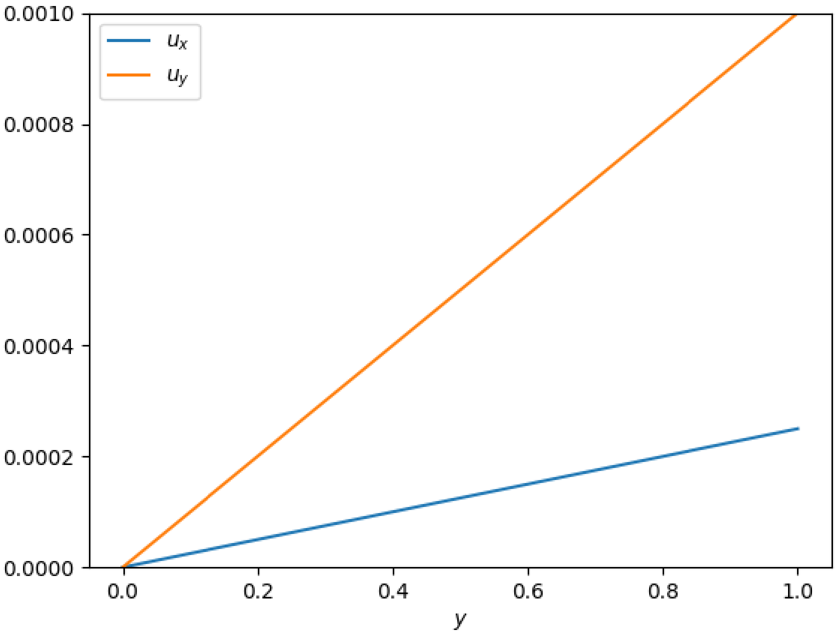


$$\sigma_{yy} = M u_{y,y}$$

$$\sigma_{xy} = G u_{x,y}$$

Shear of an infinite strip - Cauchy

- Open and run file 001.py
- Is this the expected result?
- How is the result afftected by increasing Poisson ratio?



Shear of an infinite strip - Cosserat

- The internal virtual power reads

$$P_{int} = \sigma_{yy}\gamma_{yy} + \sigma_{xy}\gamma_{xy} + \sigma_{yx}\gamma_{yx} + \mu_{zy}\kappa_{zy}$$

- The geometry and stresses remain the same
- Zero moment at the upper boundary
- ω are the Cosserat rotations and μ the couple stresses
- γ are the generalized strains and κ the curvatures
- η1 = 0.8 , η3 = 2.5, l =0.67 m

$$\sigma_{yy} = M u_{y,y}$$

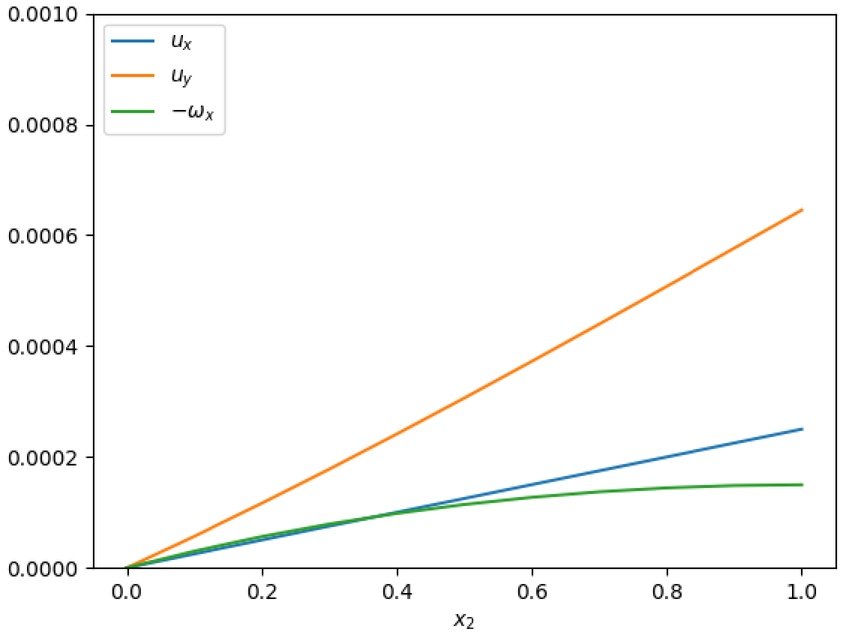
$$\sigma_{xy} = G(1+\eta_1)u_{x,y} + 2G\eta_1\omega_z$$

$$\sigma_{yx} = G(1-\eta_1)u_{x,y} - 2G\eta_1\omega_z$$

$$\mu_{zy} = 2\eta_3 G l^2 \omega_{z,y}$$

Shear of an infinite strip - Cosserat

- Open and run file 002.py
- How is the result different?
- Is the stored energy higher or lower?
- How is the result affected by increasing Poisson ratio?



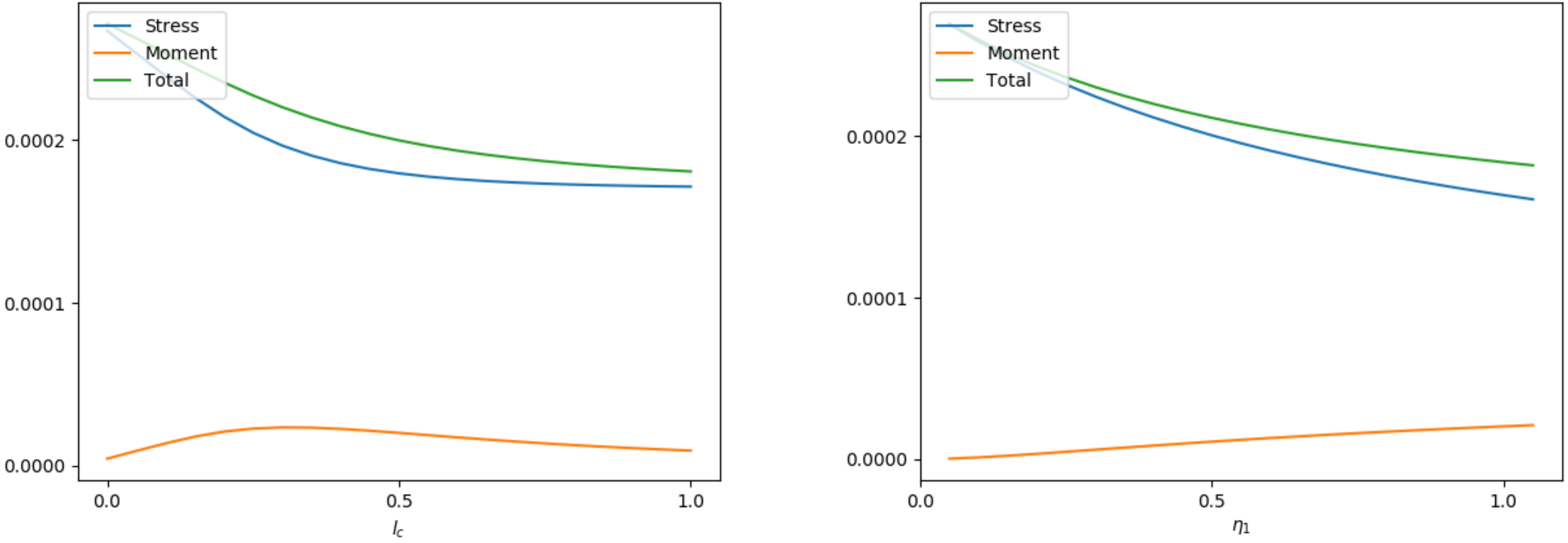
Shear of an infinite strip - Cosserat

- Save file 002.py as 003.py
- Run a parametric analysis to assess the effect of the characteristic length on the stored energy and plot the results
  - You can use the time dependent solution of the last session as an aid

- Modify to assess the effect of η_1 on the stored energy and plot the results

- Given the results, is the safety assessment for a beam with given load at the tip on the safe side with a Cosserat or a Cauchy constitutive law?

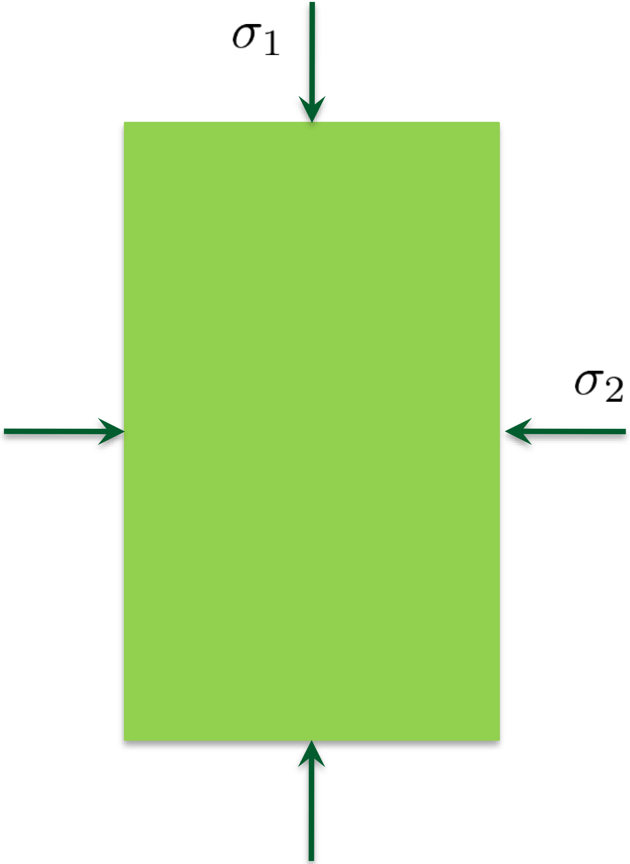
Shear of an infinite strip – Results of 003.py – stress boundary



Anisotropic elasticity

- The biaxial compression of a rectangular specimen is considered.
- The height is 0.2 m, the width 0.1 m

- Stress boundary conditions with:
  - Vertical displacement constrained at the bottom boundary
  - Horizontal displacement constrained at the midpoint of the bottom boundary



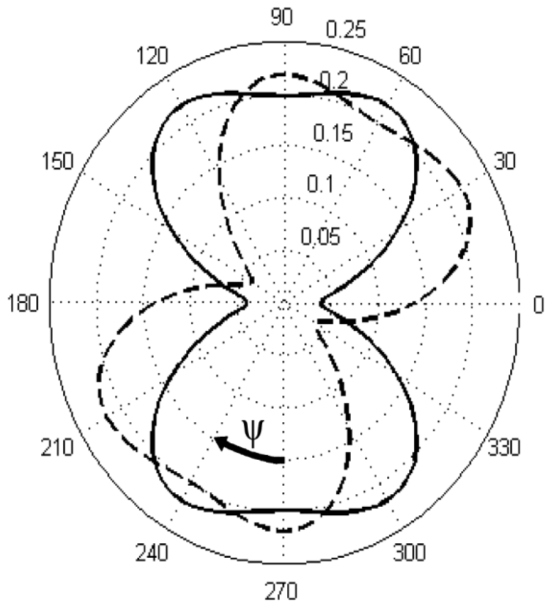
Anisotropic elasticity

- Constitutive law

$$p(\theta) = \frac{1 - c\cos\left(2(\theta+\psi)\right) - d\cos\left(4(\theta+\psi)\right)}{2\pi}$$

- After Gerolymatou (2014) for a Cosserat continuum

- Constrained to yield a symmetric stress tensor



Anisotropic elasticity

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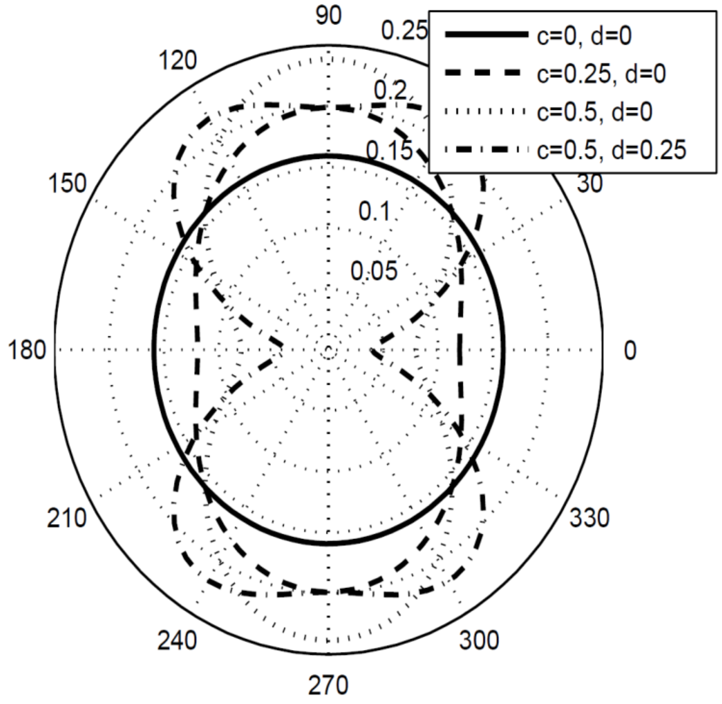
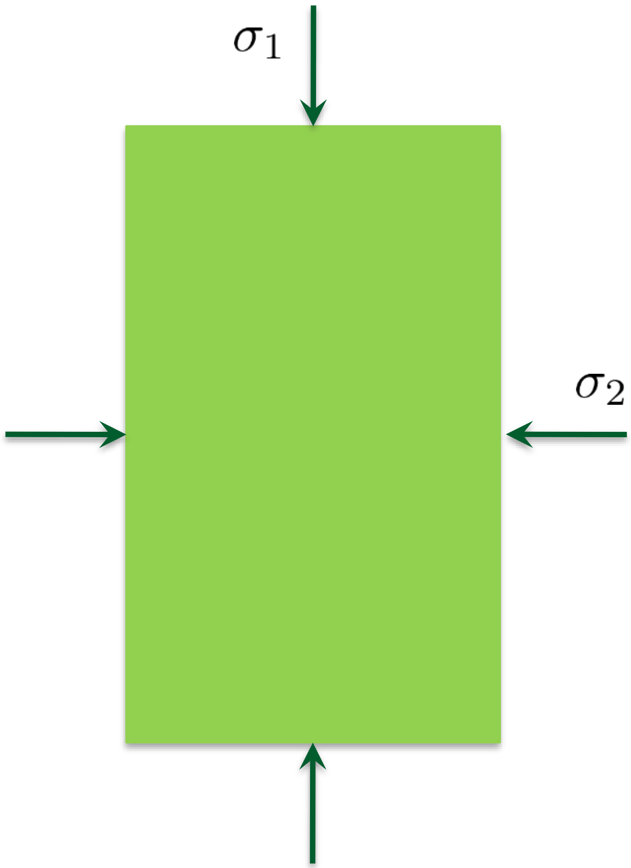


Anisotropic elasticity

- Open file 004.py and run it
- Is the result as expected?

- Modify the parameters of the stiffness:
  - Shear modulus and Poisson ratio
  - Anisotropy and direction
- What happens to the stored energy?

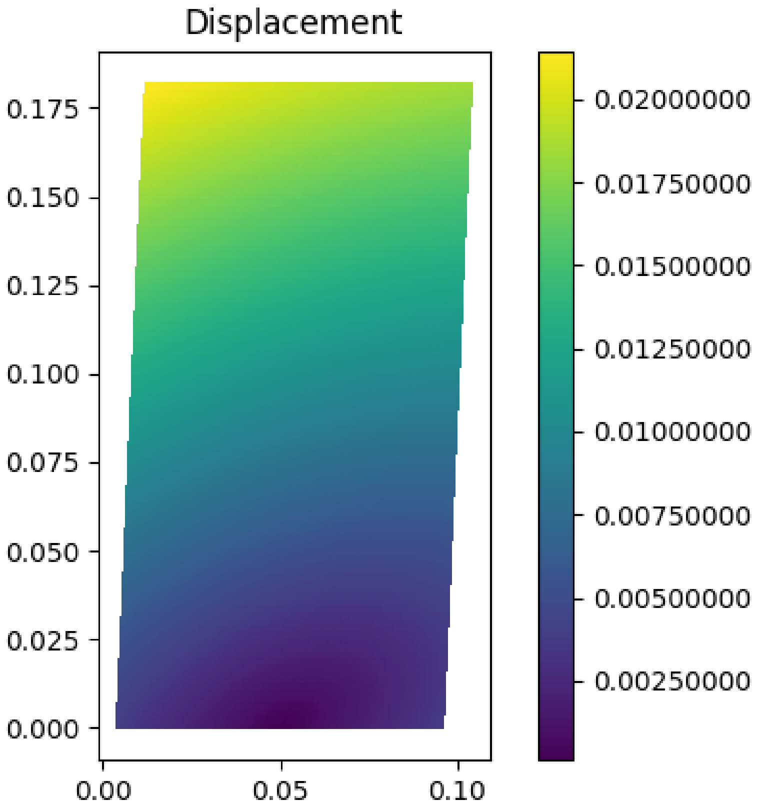
- What happens to the displacements?



Anisotropic elasticity - results

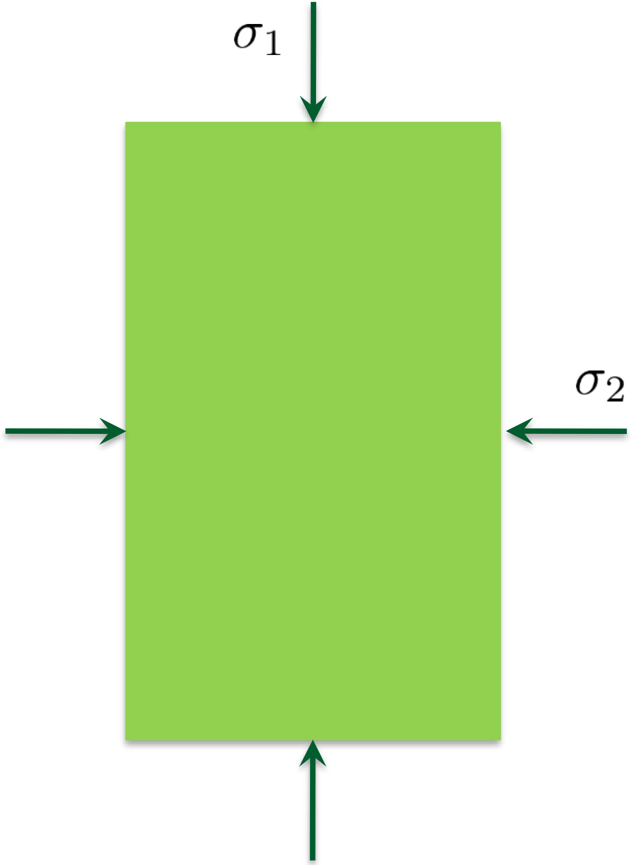
$$c = 0.5$$

$$\phi = \pi/6$$

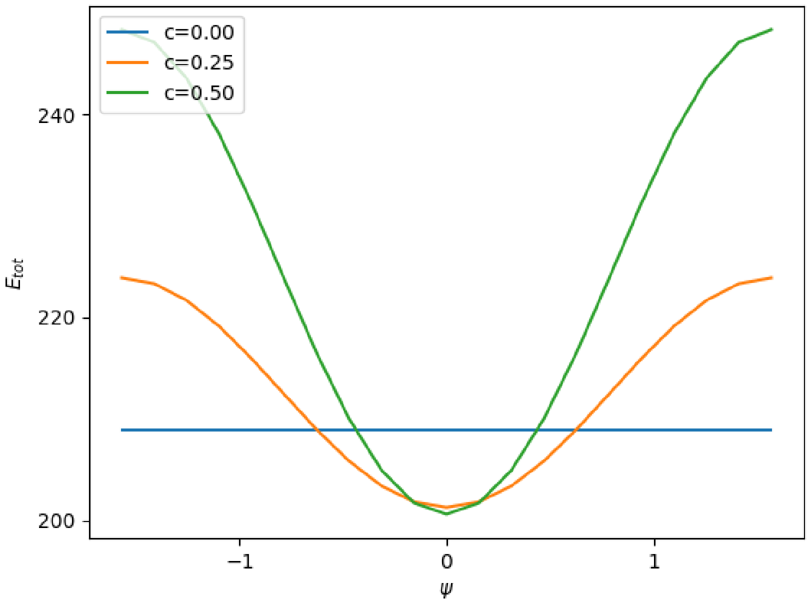


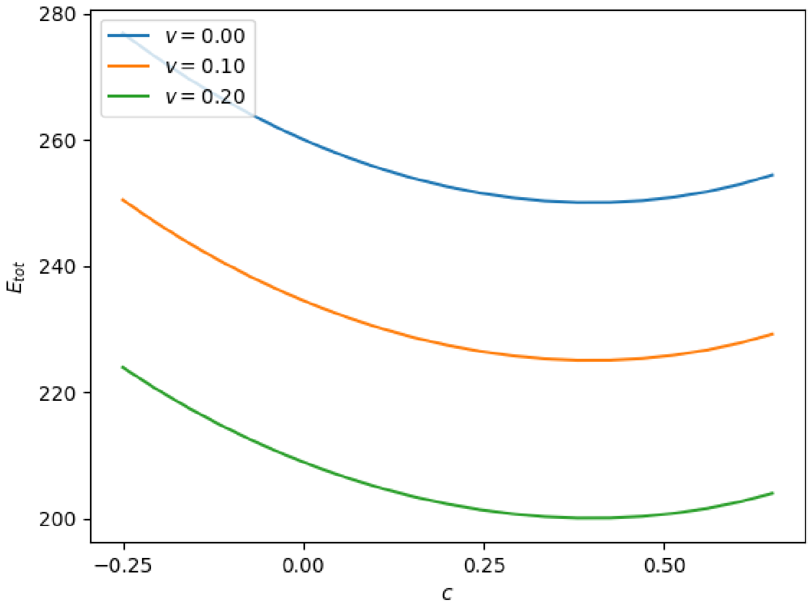
Anisotropic elasticity

- Save the script as 005.py
- Modify the script to perform a parametric analysis
- Test the effect of the parameters:
  - c and ψ
- For the location of the energy minimum test the effect of the parameters
  - Shear modulus
  - Poisson ratio
- Which factor is more significant?

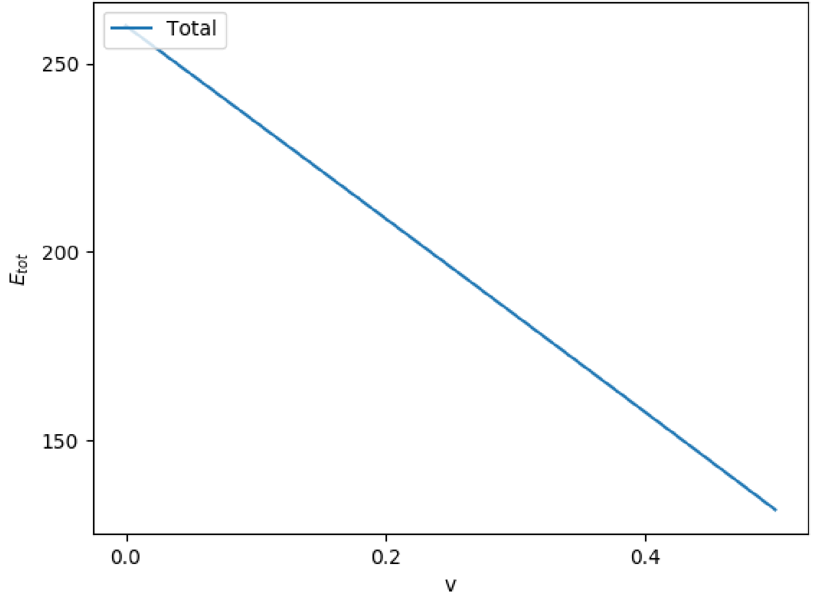


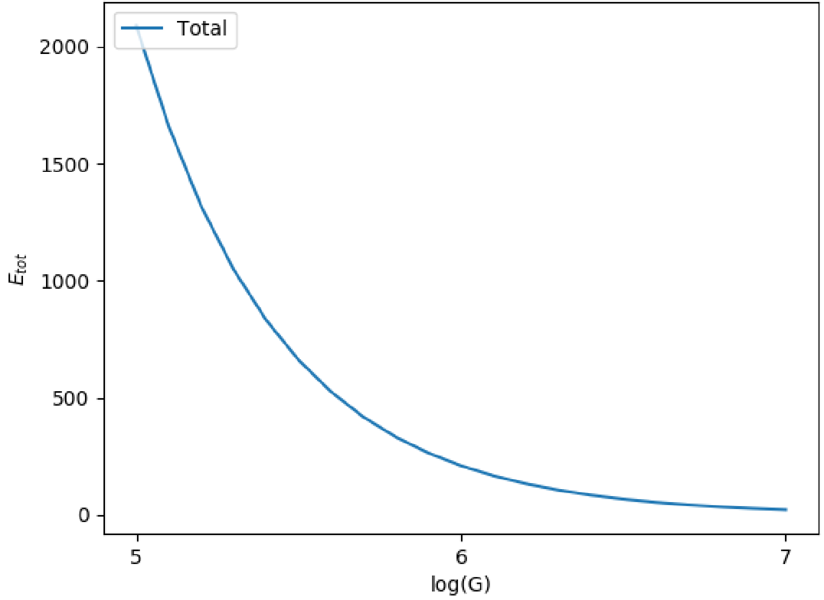
Anisotropic elasticity - results





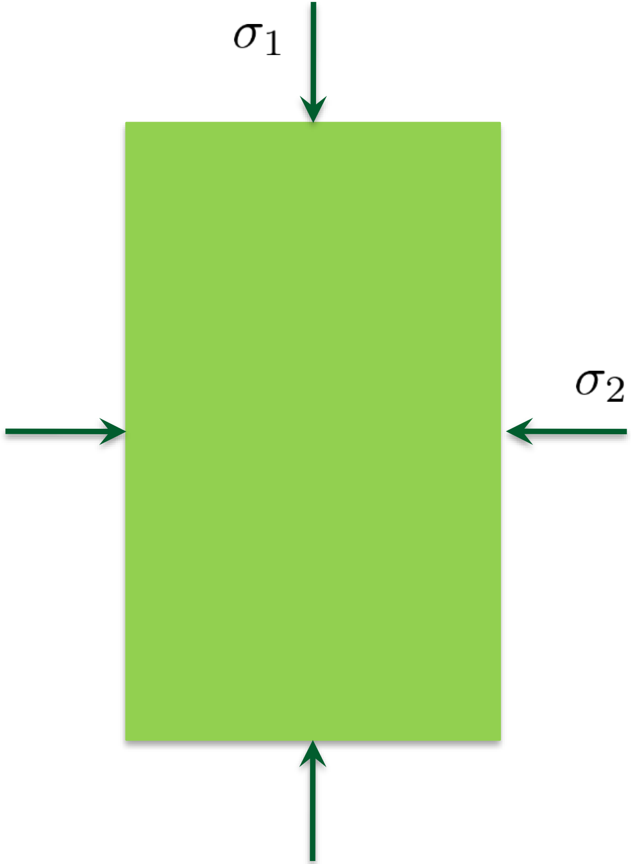
Anisotropic elasticity - results



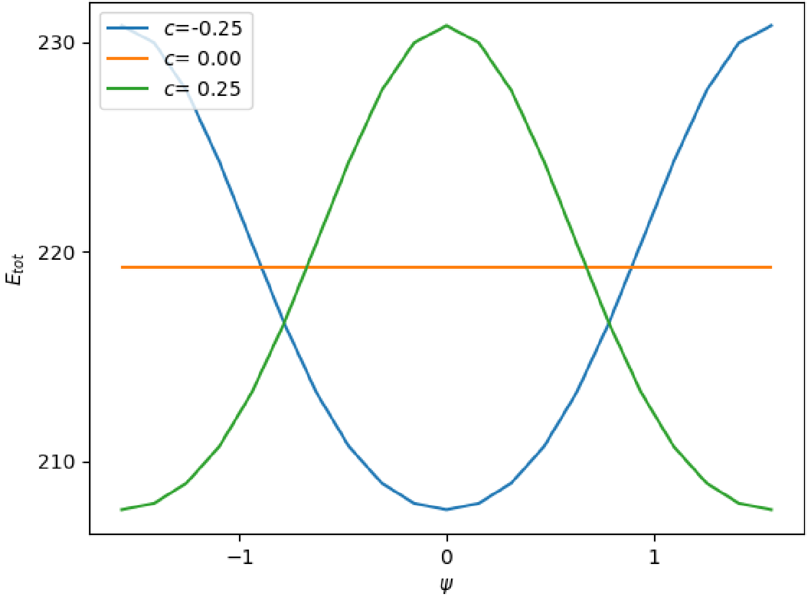


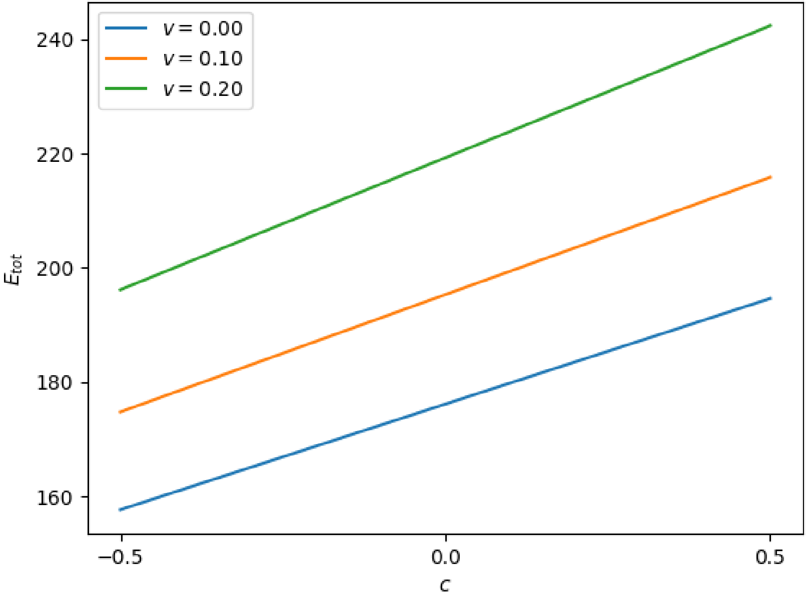
Anisotropic elasticity

- Save the script as 006.py
- Perform the same analysis for displacement controlled boundaries.
- Set the vertical displacement to 0.02 and the horizontal displacement to plus/minus 0.0032
- Where is the minimum with respect to ψ located now?

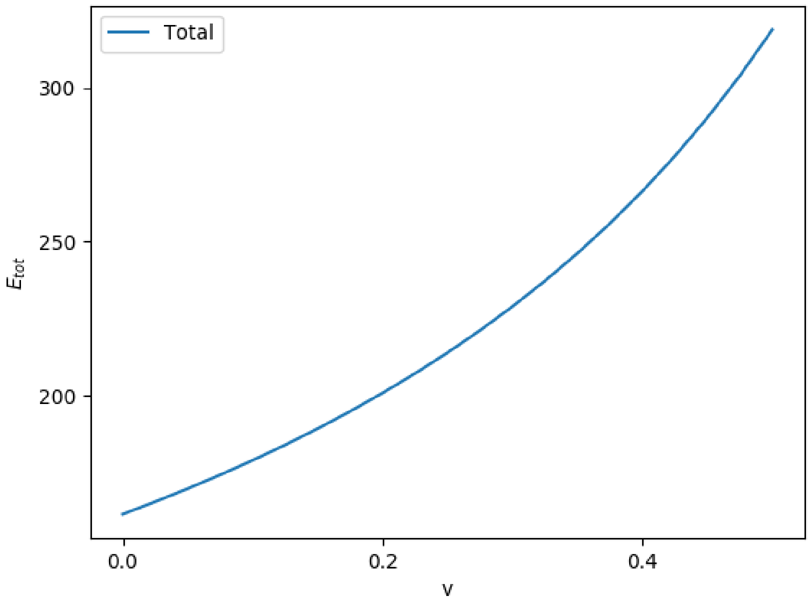


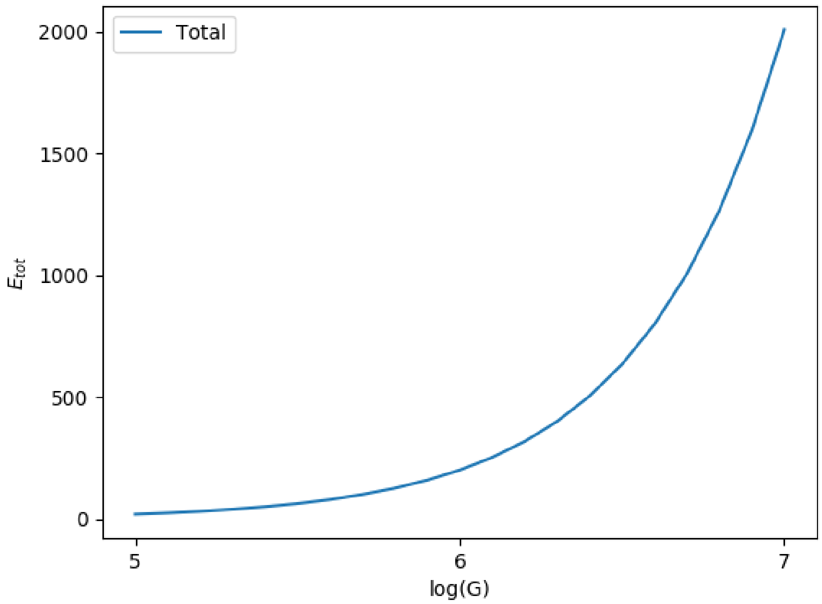
Anisotropic elasticity - results





Anisotropic elasticity - results





Elasto-plasticity

- A Mohr-Coulomb criterion is assumed:

$$f = q - \sin(\phi)p + 2c\cos(\phi)$$

- where in principal directions

$$p = \frac{\sigma_1 + \sigma_2}{2}, \quad q = \frac{\sigma_1 - \sigma_2}{2}, \quad \epsilon_p = \frac{\epsilon_1 + \epsilon_2}{2}, \quad \epsilon_q = \frac{\epsilon_1 - \epsilon_2}{2}$$

- We assume the usual:

$$\dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^{el} + \dot{\epsilon}_{ij}^{pl}$$

- and, to keep things simple:

$$c = const, \quad \phi = const$$

Elasto-plasticity

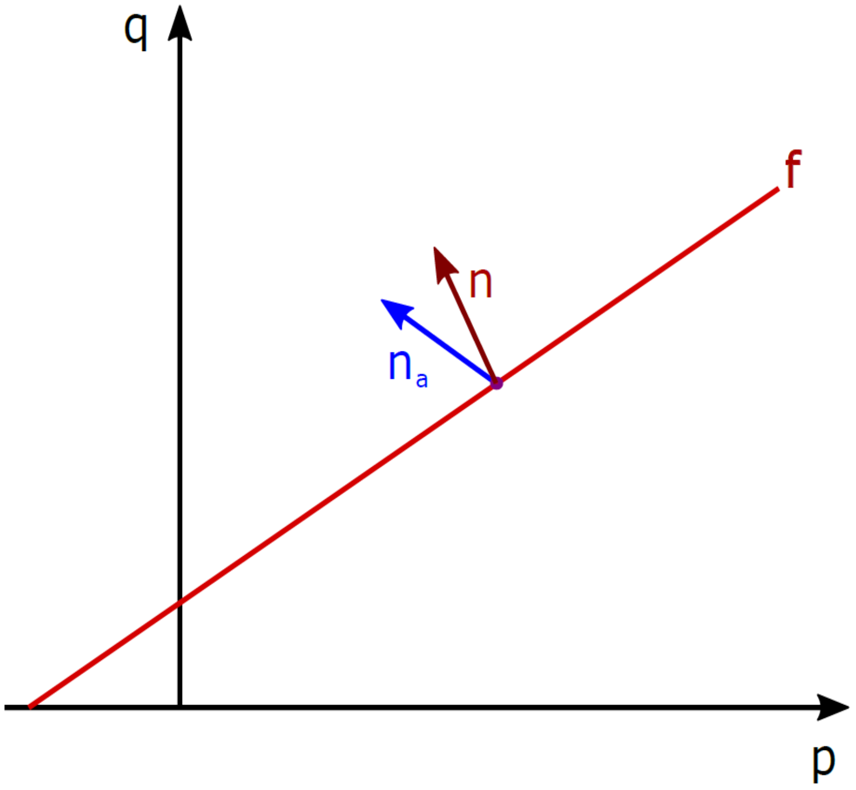
- The flow rule reads: $\dot{\epsilon}_{ij}^{pl} = \dot{\lambda}\frac{\partial g}{\partial \sigma_{ij}}$
- In the p-q space this simplifies to:

$$\left(\dot{\epsilon}_p^{pl}, \dot{\epsilon}_q^{pl}\right) = \dot{\lambda}\left(1, -\sin(\phi + a)\right) = \dot{\lambda}\mathbf{n}_a$$

- For a plastic step, the yield surface and its derivative should also be zero.
- Solving the conditions to satisfy for $\dot{\lambda}$

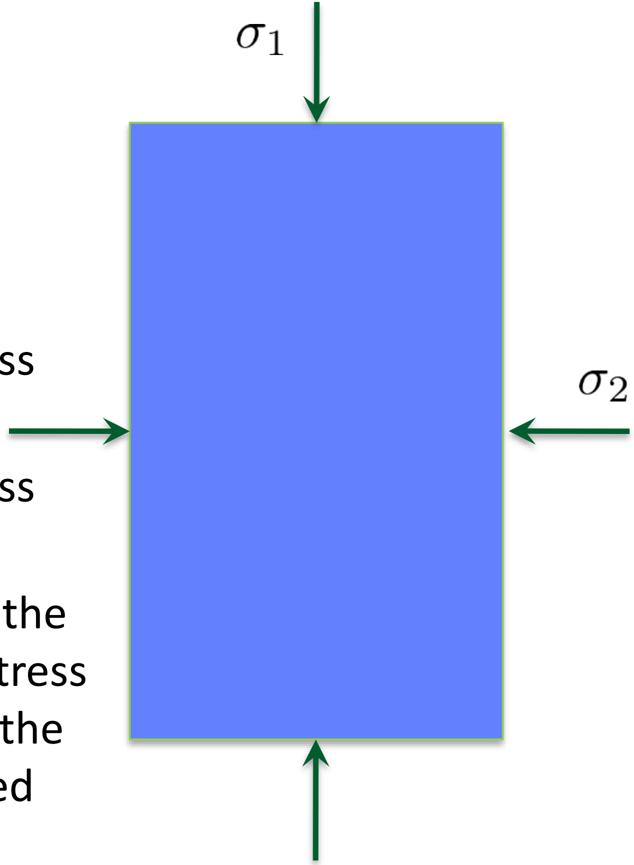
$$2\left(\mu + (\mu + \lambda)\sin(\phi)\sin(\phi + a)\right)\dot{\lambda} = f_{trial}$$

- where μ, λ are elastic constants.



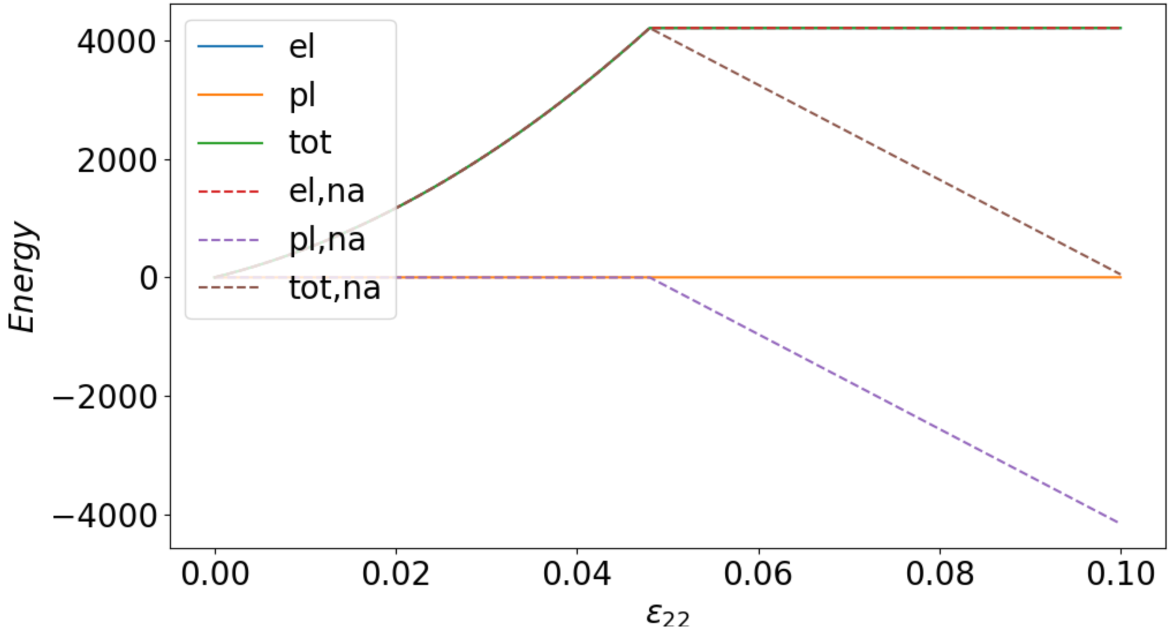
Elasto-plasticity

- Open file 007.py and let's walk through it.
- FEniCS has not been used for this example.
- It solves the problem for one material point.
    - The vertical strain increment and the horizontal stress are known
    - The horizontal strain increment and the vertical stress can be evaluated for elasticity
    - If the new stress state lies outside the yield surface, the vertical plastic strain increment and the horizontal stress are known. The vertical stress is known, as it lies on the yield surface. The horizontal plastic strain is evaluated from the flow rule.



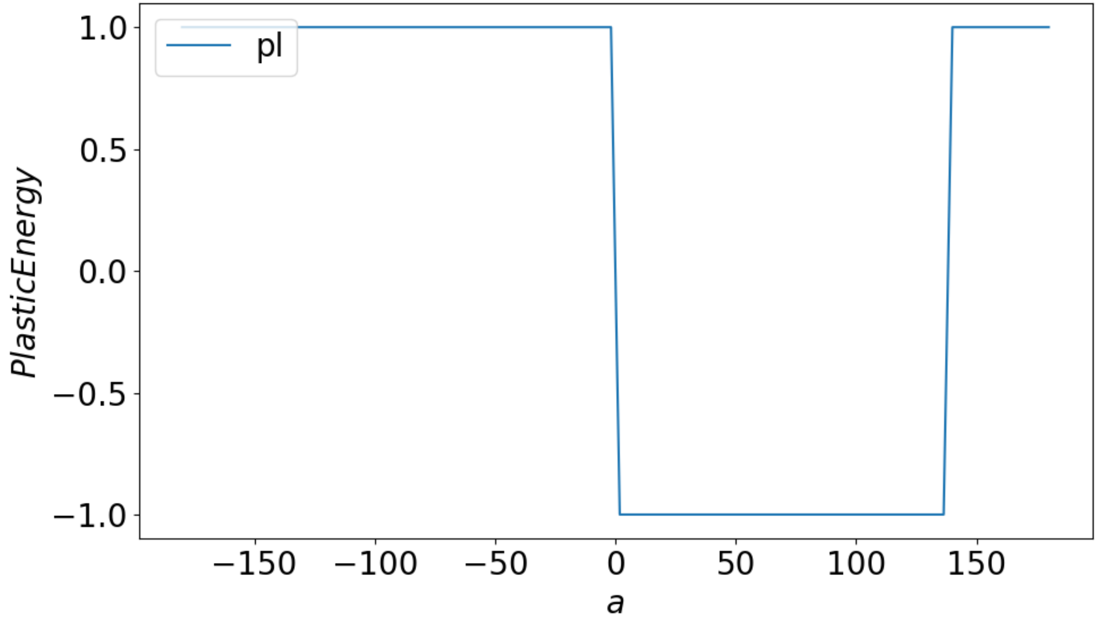
Elasto-plasticity

- Run the code for associative and non – associative flow rule.



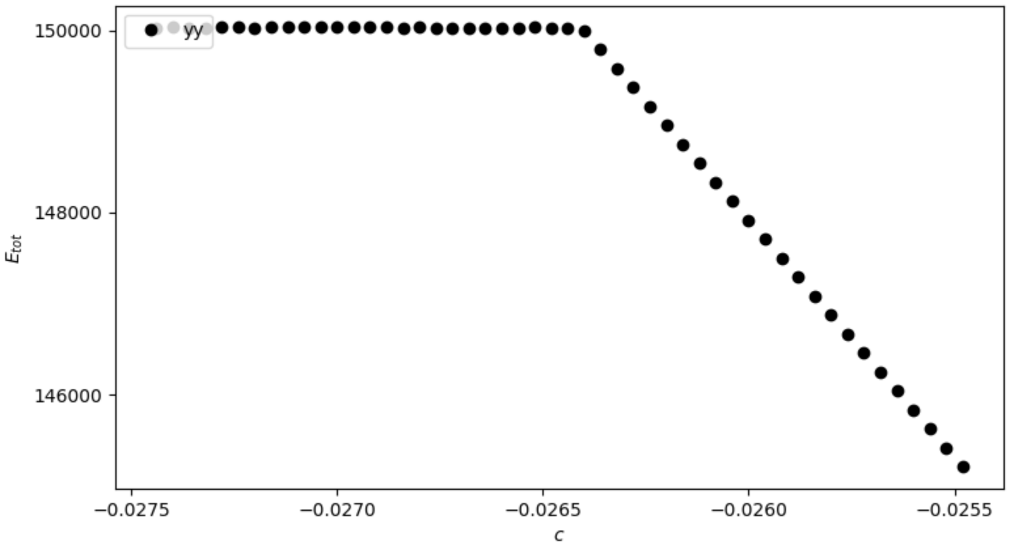
Elasto-plasticity

- Are sign changes likely? Try file 007b.py – Attention, it‘s slow!



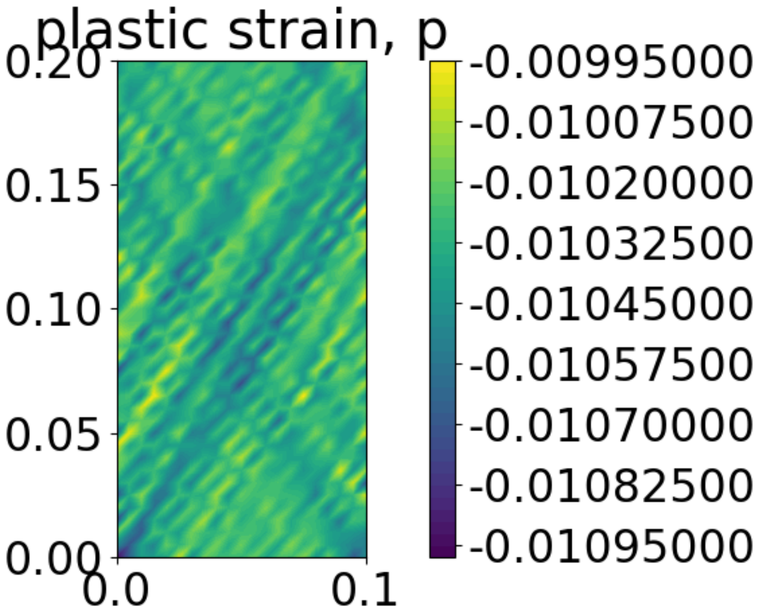
Elasto-plasticity

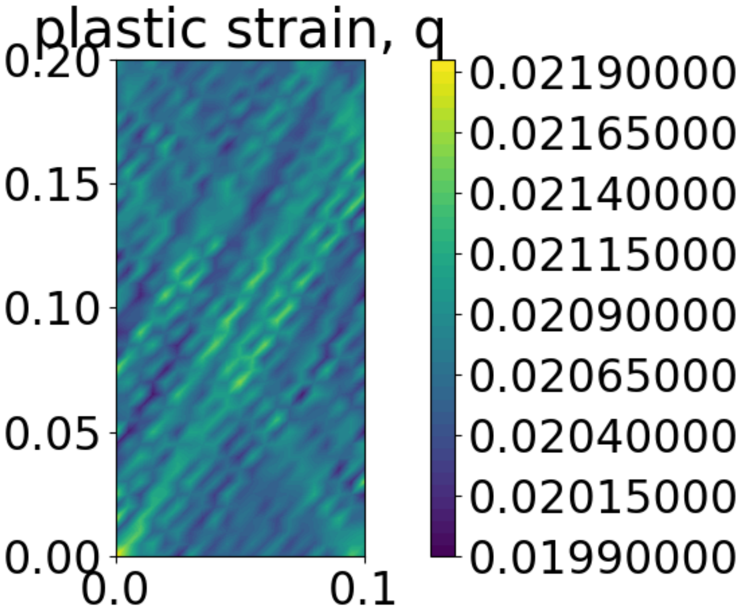
- Alternatively, using FEniCS: try file 007d.py



Elasto-plasticity

- Slow convergence, mostly due to perfect plasticity. Does the problem persist for an oedometric test?





Thermal diffusion

- The general form of the heat equation reads:

$$\rho c_p \frac{\partial T}{\partial t} - \nabla.(k\nabla T) = \dot{q}$$

  - ρ is the density
  - cp the specific heat capacity
  - k the thermal conductivity
  - $\dot{q}$ the volumetric heat source
  - T the temperature
- To simplify things it is assumed that there is no volumetric heat sourse and that the thermal conductivity is constant:

$$\rho c_p \frac{\partial T}{\partial t} - k\nabla^2 T = 0$$

Thermal diffusion

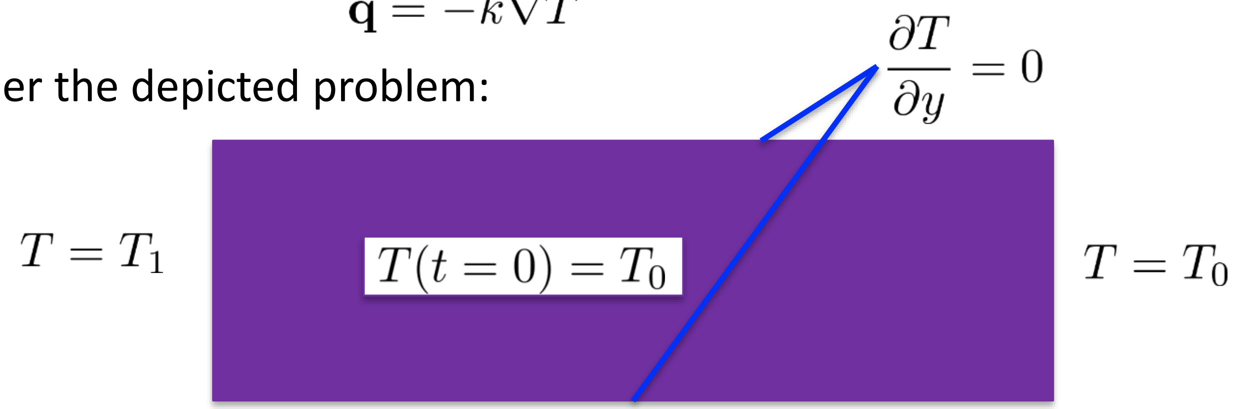
- Chosing zero heat energy zero temperature, the heat energy reads:

$$Q = \rho c_p T$$

- while the rate of flow of heat energy per unit area through a surface is proportional to the negative temperature gradient across the surface:

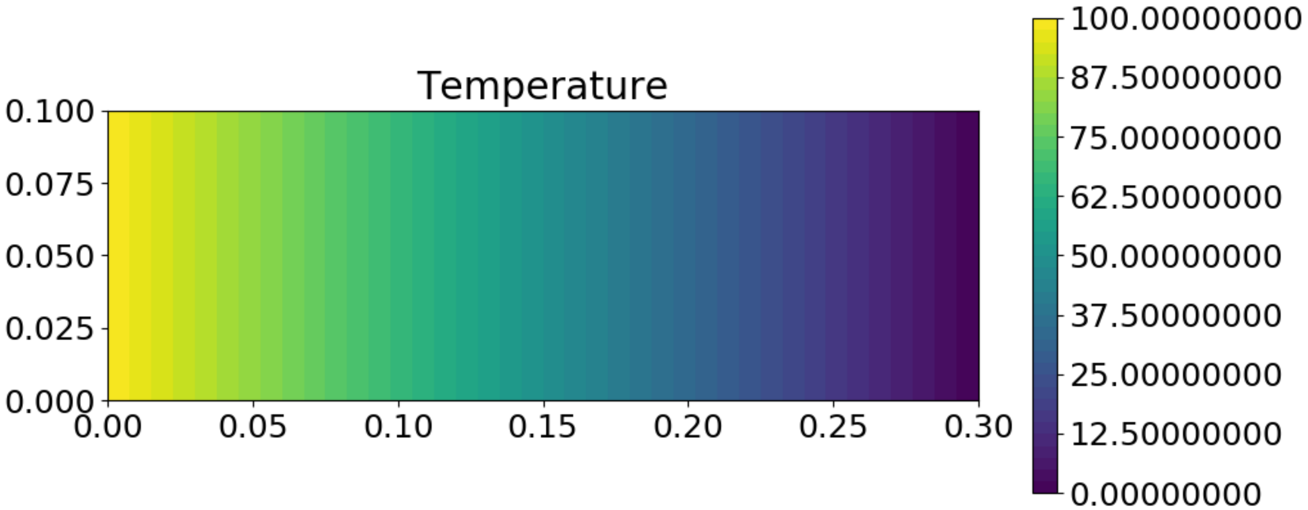
$$\mathbf{q} = -k \nabla T$$

- We consider the depicted problem:



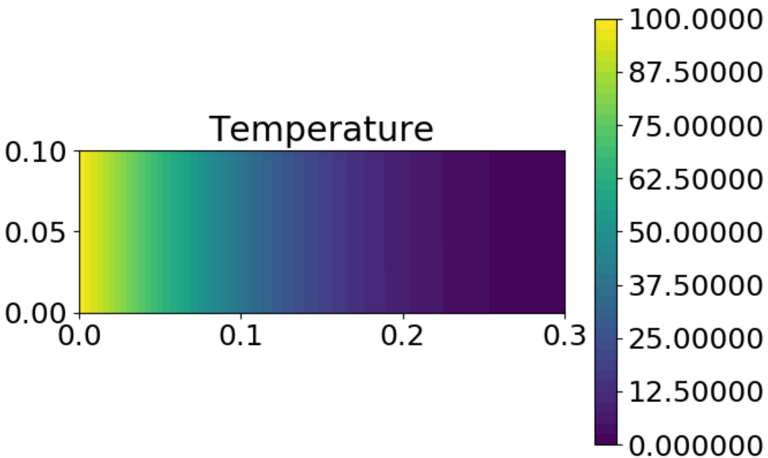
Thermal diffusion

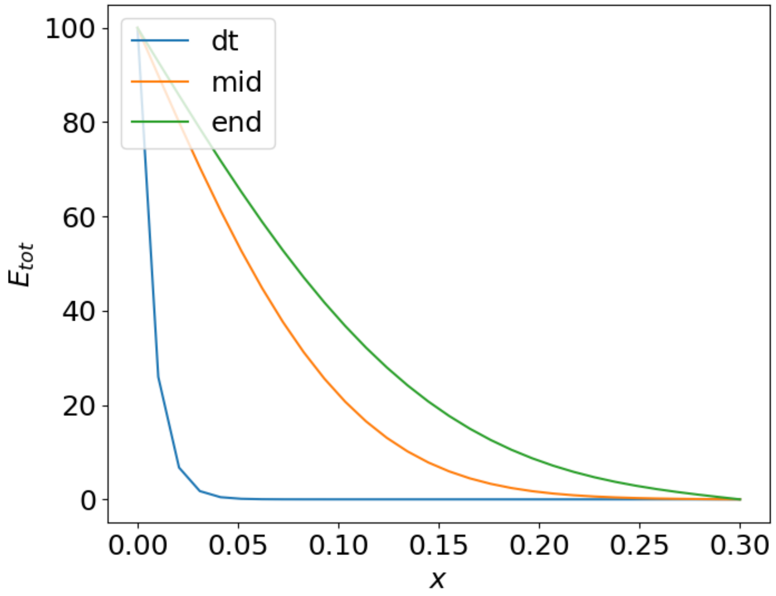
- Open file 008.py
- It solves the heat diffusion equation in the steady state



Thermal diffusion

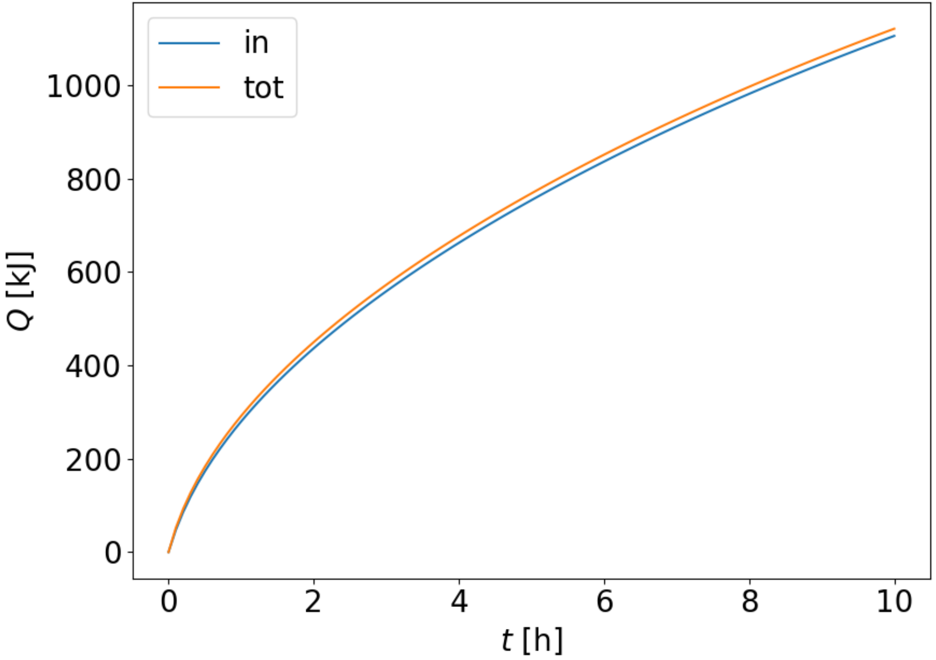
- Open file 009.py
- Fill in what is missing, so that it solves the transient heat diffusion equation.





Thermal diffusion

- Plot the evolution of the thermal energy.



Thermal diffusion – thermal strains

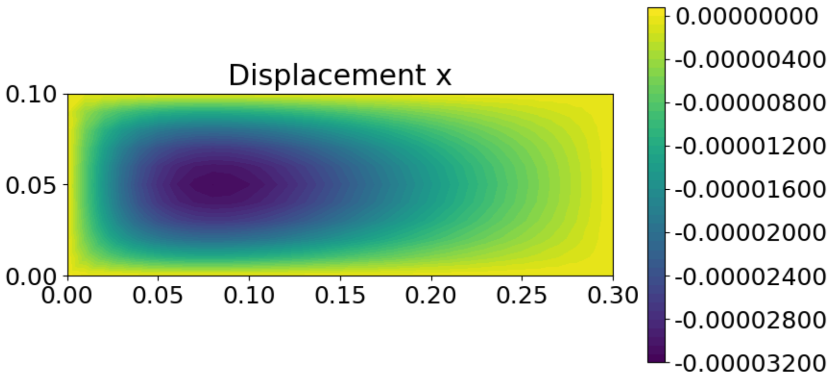
- Most materials expand as a result of increasing temperature. This leads to thermal strains:

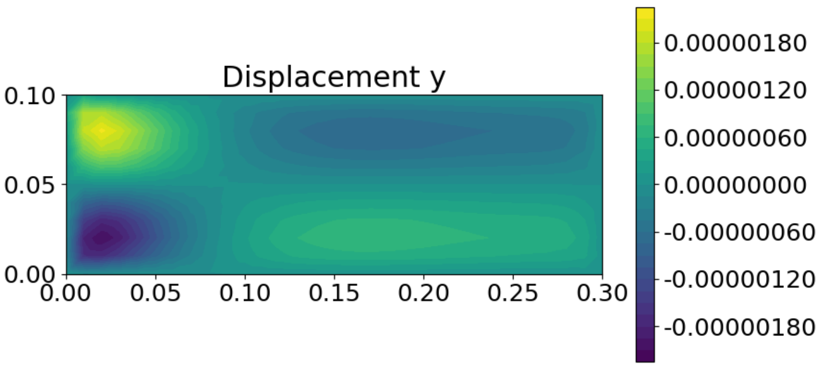
$$\epsilon_{ij}^{T} = -\alpha T \delta_{ij}$$

  - a is the thermal expansion coefficient
  - T is the differential temperature with respect to the initial one and
  - δ is the Kronecher delta
- The negative sign indicates expansion.
- Other laws are available, but the simplest one is assumed here.
  - no dependence of a on temperature
  - linear response
  - no anisotropy
  - no change in the elastic material parameters

Thermal diffusion – thermal strains

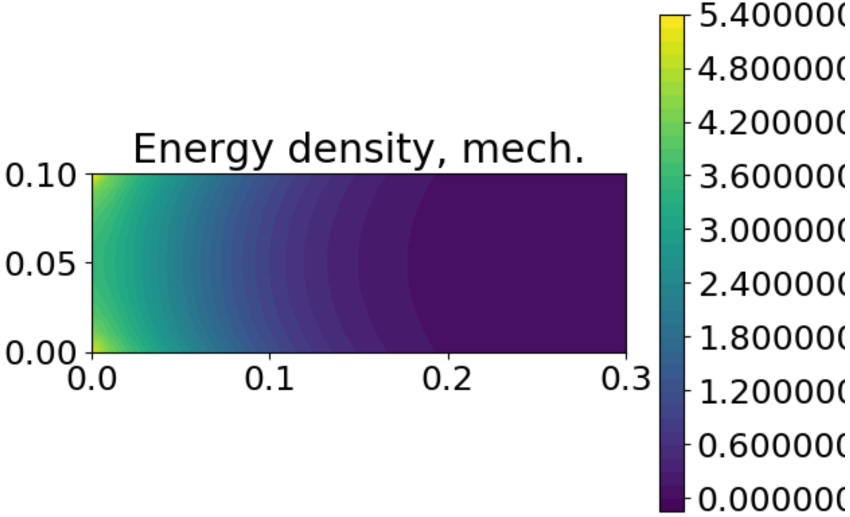
- Save the file you have been working on as 010.py and introduce the thermal strains.
- Assume a value of $2 \cdot 10^{-5}$ K^{-1} for the expansion coefficient.

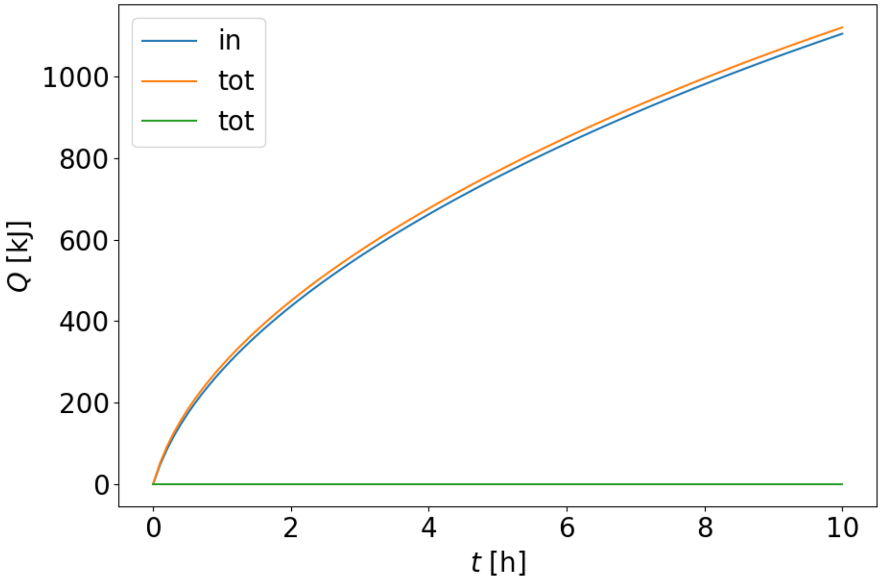




Thermal diffusion – thermal strains

- What happens to the energy? Plot elastic and thermal energy maps.





Thermal diffusion – thermal strains

- And if we change the expansion coefficient by 3 orders of magnitude?